Logistic Growth

Section 6.3A

Calculus BC AP/Dual, Revised ©2017

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RECALL

Solve the logistic differential equation $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

$$\frac{1}{P\left(1 - \frac{P}{L}\right)}dP = kdt$$

$$\int\left(\frac{1}{P} + \frac{1}{L-P}\right)dP = \int kdt$$

RECALL

Solve the logistic differential equation $\frac{dy}{dt} = kP\left(1 - \frac{P}{L}\right)$.

$$\ln|P| + \ln|L - P| = kt + C$$

$$\ln\left|\frac{L - P}{P}\right| = -kt - C$$

$$\left|\frac{L - P}{P}\right| = e^{-kt-C}$$

$$\frac{L - P}{P} = e^{-C}e^{-kt}$$

$$\frac{L - P}{P} = Be^{-kt}$$

$$P = \frac{L}{1 + Be^{-kt}}$$

LOGISTIC CURVES

A. Logistics Differential Equation: $\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$ or $\frac{dP}{dt} = kP(L - P)$ where k will be multiplied and the inside is divided

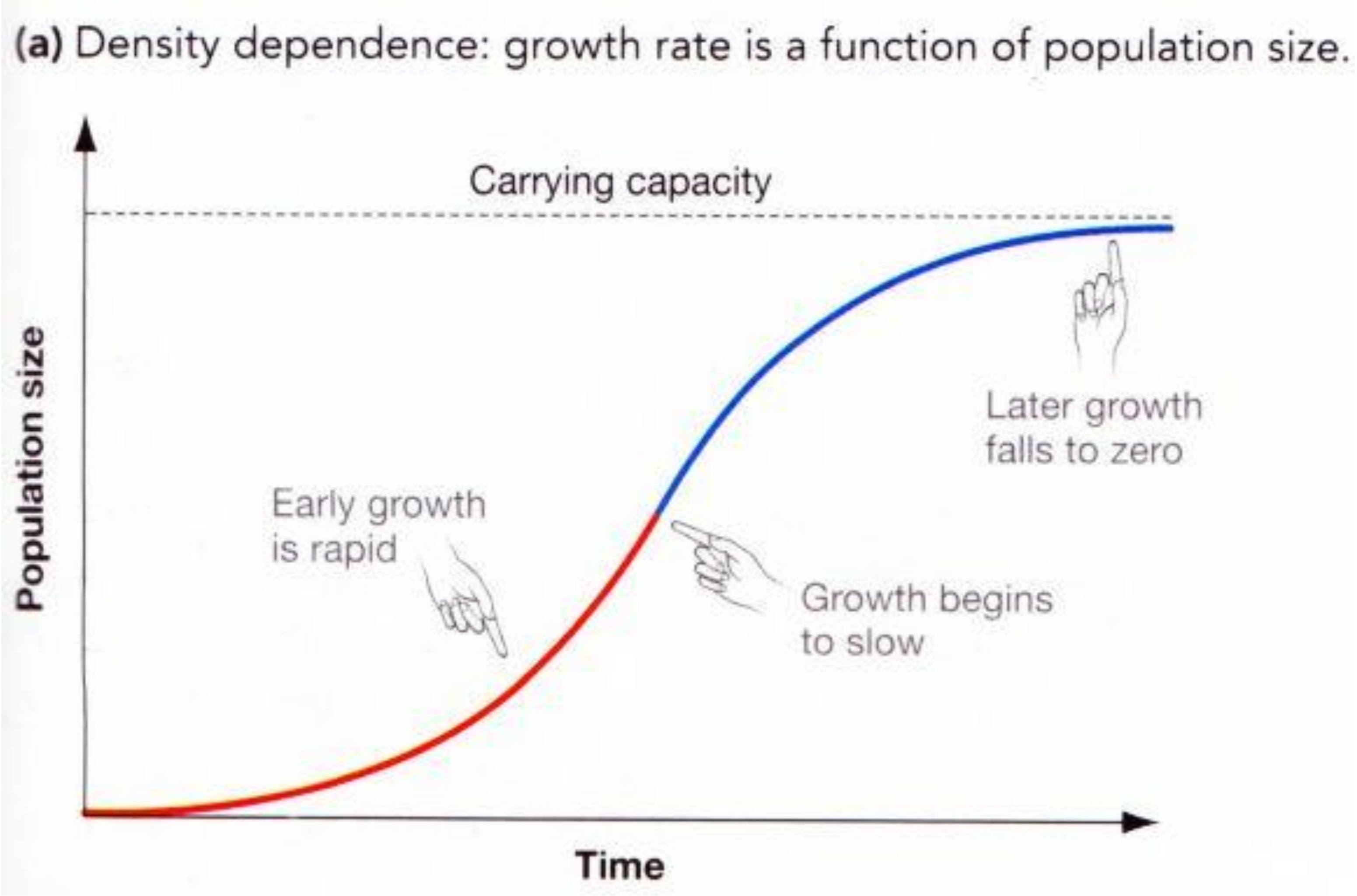
B. The k can be different depending on the use of the equation

C. Logical Growth Model: $P = \frac{L}{1+Be^{-kt}}$

1. L = Carrying Capacity (Upper Horizontal LIMIT)
2. K = Proportionality Constant
3. B = Beginning Amount (arbitrary), use the equation: $B = \frac{L-P_0}{P_0}$ where P_0 is the initial population

C. Point of Inflection: $y = \frac{L}{2}$

Logistic Growth



EXAMPLE 1

Given $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$ identify the k and L of the equation.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

$$\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$$

Example 1

Given $\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$ identify the k and L of the equation.

$$\frac{dP}{dt} = P\left(2 - \frac{P}{5000}\right)$$

$$\frac{dP}{dt} = (2)P\left(\frac{2}{2} - \frac{\frac{P}{5000}}{2}\right)$$

$$\frac{dP}{dt} = 2P\left(1 - \frac{P}{10000}\right)$$

$$k = 2$$

$$L = 10,000$$

YOUR TURN

Given $\frac{dP}{dt} = 0.01P(100 - P)$ identify the k and L of the equation.

$k = 1$

$L = 100$

Example 2

Using the equation, $y = \frac{4}{1+2e^{-3t}}$ identify the point of inflection and sketch a graph.

$$L = 4, B = 2, and\ \ k = 3$$

$$y = 4\left(1 + 2e^{-3t}\right)^{-1}$$

$$y' = 4\left(-1\right)\left(1 + 2e^{-3t}\right)^{-2}\left(-6e^{-3t}\right)$$

$$y' = 3\left(\frac{4}{1+2e^{-3t}}\right)\left(\frac{2e^{-3t}}{1+2e^{-3t}}\right)$$

$$y' = 3y\left(1 - \frac{1}{1+2e^{-3t}}\right)$$

using Long Division

Example 2

Using the equation, $y = \frac{4}{1+2e^{-3t}}$ **identify the point of inflection and sketch a graph.**

$$y' = 3y\left(1 - \frac{1}{1+2e^{-3t}}\right)$$

using Long Division

$$y = \frac{4}{1+2e^{-3(0)}}; y(0) = \frac{4}{3}$$

$$y' = 3y\left(1 - \frac{4}{4\left(1+2e^{-3t}\right)}\right)$$

$$y' = 3y\left(1 - \frac{y}{4}\right)$$

EXAMPLE 2

Using the equation, $y = \frac{4}{1+2e^{-3t}}$ identify the point of inflection and sketch a graph.

$$y = \frac{L}{2}$$

$$y = \frac{4}{2}$$

$$y = 2$$

$$y = \frac{4}{1+2e^{-3t}}$$

$$2 = \frac{4}{1+2e^{-3t}}$$

$$2\left(1+2e^{-3t}\right) = 4$$

$$1+2e^{-3t} = 2$$

$$2e^{-3t} = 1$$

$$e^{-3t} = \frac{1}{2}$$

$$\frac{1}{e^{3t}} = \frac{1}{2}$$

$$e^{3t} = 2$$

$$\ln e^{3t} = \ln 2$$

$$3t = \ln 2$$

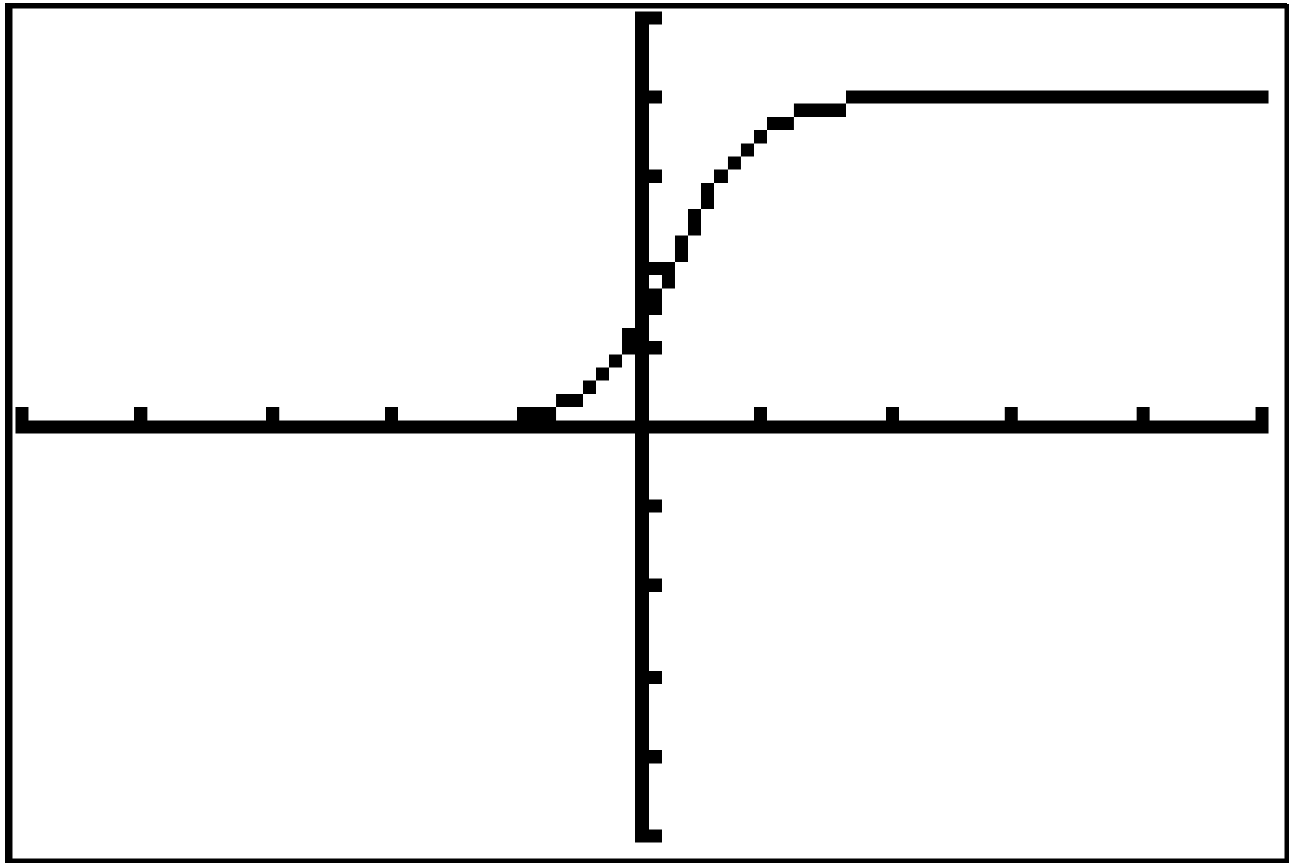
$$t = \frac{\ln 2}{3}$$

$$POI : \left(\frac{\ln 2}{3}, 2\right)$$

EXAMPLE 2

Using the equation, $y = \frac{4}{1+2e^{-3t}}$ identify the point of inflection and sketch a graph.

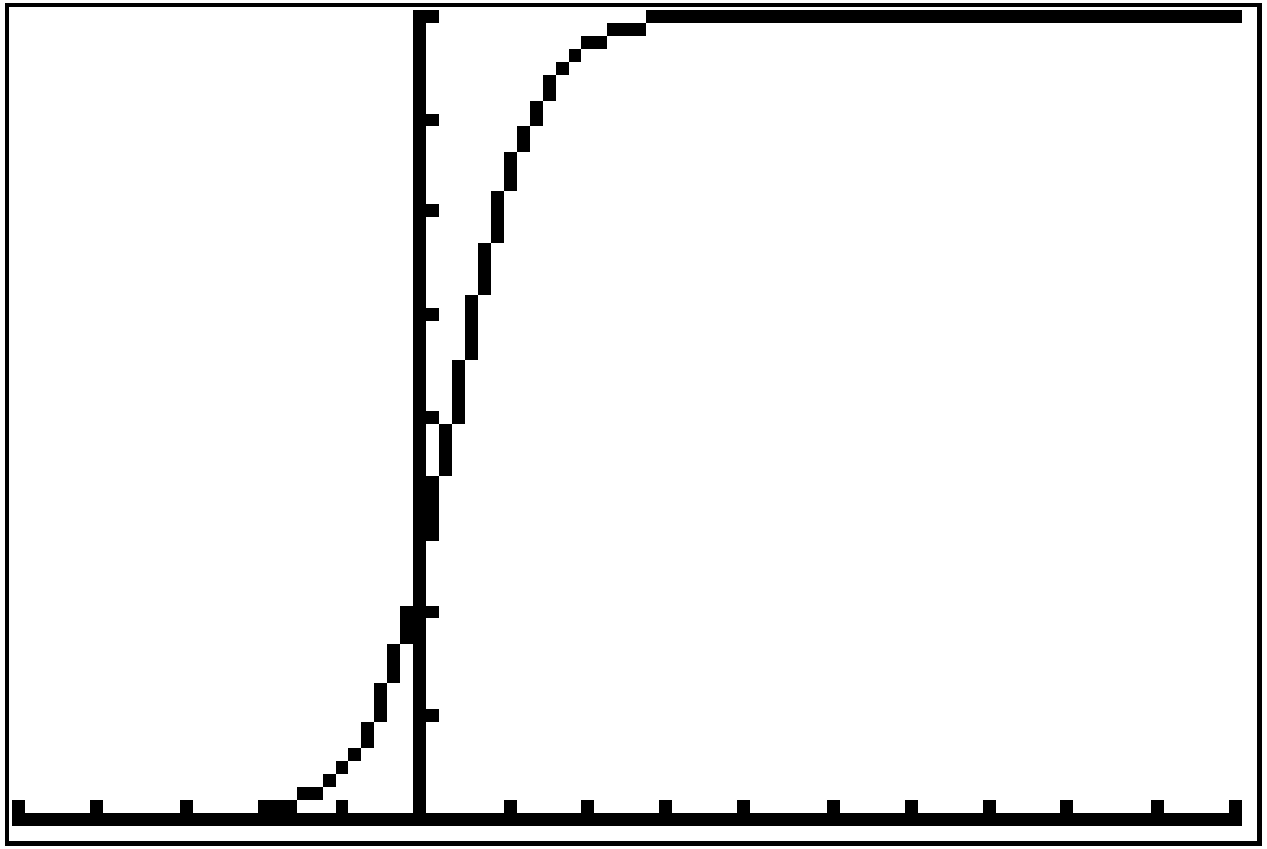
$$POI: \left(\frac{\ln 2}{3}, 2\right)$$



YOUR TURN

Using the equation, $y = \frac{8}{1+2e^{-2t}}$ **identify the point of inflection and sketch a graph.**

$$POI:\left(\frac{\ln 2}{2}, 4\right)$$



EXAMPLE 3

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population, p, is: $\frac{dP}{dT} = kP\left(1 - \frac{P}{4000}\right)$, $40 \le p \le 4000$ where t is the number of years.

A. Write a model for the elk population in terms of t.
B. Estimate the elk population in 15 years.
C. Find the limit of the model as $t \to \infty$.

EXAMPLE 3A

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population, $\boldsymbol{p}$, is: $\frac{dP}{dT} = kP\left(1 - \frac{P}{4000}\right)$, $40 \le p \le 4000$ where $\boldsymbol{t}$ is the number of years.

$$40 = \frac{4000}{1 + Be^{-k(0)}}$$

A. Write a model for the elk population in terms of $\boldsymbol{t}$.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{4000}\right), 40 \le P \le 4000$$

$$y = \frac{L}{1 + Be^{-kt}}$$

$$40\left(1 + Be^{-k(0)}\right) = 4000$$

$$\mathrm{L} = 4000, k = ??, \mathrm{B} = 99$$

$$P = \frac{4000}{1 + Be^{-kt}}$$

$$40 + 40B = 4000$$

$$P = \frac{4000}{1 + 99e^{-kt}}$$

$$40B = 3960$$

$$B = 99$$

Example 3A

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population, *p*, is: $\frac{dP}{dT} = kP\left(1 - \frac{P}{4000}\right)$, $40 \le p \le 4000$ where *t* is the number of years.

A. Write a model for the elk population in terms of t.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{4000}\right), 40 \le P \le 4000$$

$$L = 4000, k \approx 0.1942, \mathrm{B} = 99$$

$$P = \frac{4000}{1+99e^{-0.1942t}}$$

$$y = \frac{4000}{1+99e^{-kt}}$$

$$104 = \frac{4000}{1+99e^{-k(5)}}$$

$$k \approx 0.1942$$

$$104\left(1+99e^{-5k}\right) = 4000$$

$$1+99e^{-5k} = \frac{4000}{104}$$

$$99e^{-5k} = \frac{4000}{104} - 1$$

$$\frac{99e^{-5k}}{99} = \frac{37.5}{99}$$

$$e^{-5k} = \frac{37.5}{99}$$

$$\frac{1}{e^{5k}} = \frac{37.5}{99}$$

$$(37.5)e^{5k} = 99$$

$$e^{5k} = 2.64$$

$$\ln e^{5k} = \ln 2.64$$

$$\frac{5k}{5} = \frac{\ln 2.64}{5}$$

EXAMPLE 3B

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population, *p*, is: $\frac{dP}{dT} = kP\left(1 - \frac{P}{4000}\right)$, $40 \le p \le 4000$ where *t* is the number of years.

B. Estimate the elk population in 15 years.

$$\frac{dP}{dt} = kp\left(1 - \frac{P}{4000}\right), 40 \le P \le 4000$$

$$L = 4000, k = ??, \mathrm{B} = 99$$

$$P = \frac{4000}{1 + 99e^{-0.1942t}}$$

$$P = \frac{4000}{1 + 99e^{-0.1942(15)}}$$

$$\approx 627.2598 \text{ elk}$$

EXAMPLE 3C

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population, *p*, is: $\frac{dP}{dT} = kP\left(1 - \frac{P}{4000}\right)$ $, 40 \le p \le 4000$ **where *t* is the number of years.**

C. Find the limit of the model as $t \to \infty$**.**

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{4000}\right), 40 \le P \le 4000$$

$$L = 4000, k = ??, \mathrm{B} = 99$$

$$P = \frac{4000}{1 + 99e^{-0.1942t}}$$

$$\lim_{t \to \infty} \frac{4000}{1 + 99e^{-0.1942(t)}}$$

4000

EXAMPLE 4

Suppose the population of bears in a national park grows according to the logistic differential equation, $\frac{dP}{dT} = 5P - 0.002P^2$, where P is the number of bears at time t in years. Find the limit of the model as $t \to \infty$ where $P(0) = 100$.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right)$$

$$\frac{dP}{dt} = 5P - 0.002P^2$$

$$\frac{dP}{dt} = \frac{5P}{5} - \frac{0.002P^2}{5}$$

$$\frac{dP}{dt} = 5P\left(1 - \frac{P}{2500}\right)$$

$$\frac{2500}{1+24e^{-0.002(t)}} = 100$$

$$B = 24$$

$$\lim_{x\to\infty} \frac{2500}{1+24e^{-0.002(t)}}$$

2500

YOUR TURN

Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears. Assuming a logistic growth model, when will the bear population reach 50? 75? 100?

$$P=\frac{L}{1+Be^{-kt}}$$

$$10=\frac{100}{1+Be^{-100(0)}}$$

$$10=\frac{100}{1+B}$$

$$(0,10)$$

$$(10,23)$$

$$10+10B=100$$

$$10B=90$$

$$B=9$$

$$P=\frac{100}{1+9e^{-kt}}$$

YOUR TURN

Ten grizzly bears were introduced to a national park 10 years ago. There are 23 bears in the park at the present time. The park can support a maximum of 100 bears. Assuming a logistic growth model, when will the bear population reach 50? 75? 100?

$$(0,10) \qquad (10,23)$$

$$23 = \frac{100}{1+9e^{-k(10)}}$$

$$1+9e^{-10k} = \frac{100}{23}$$

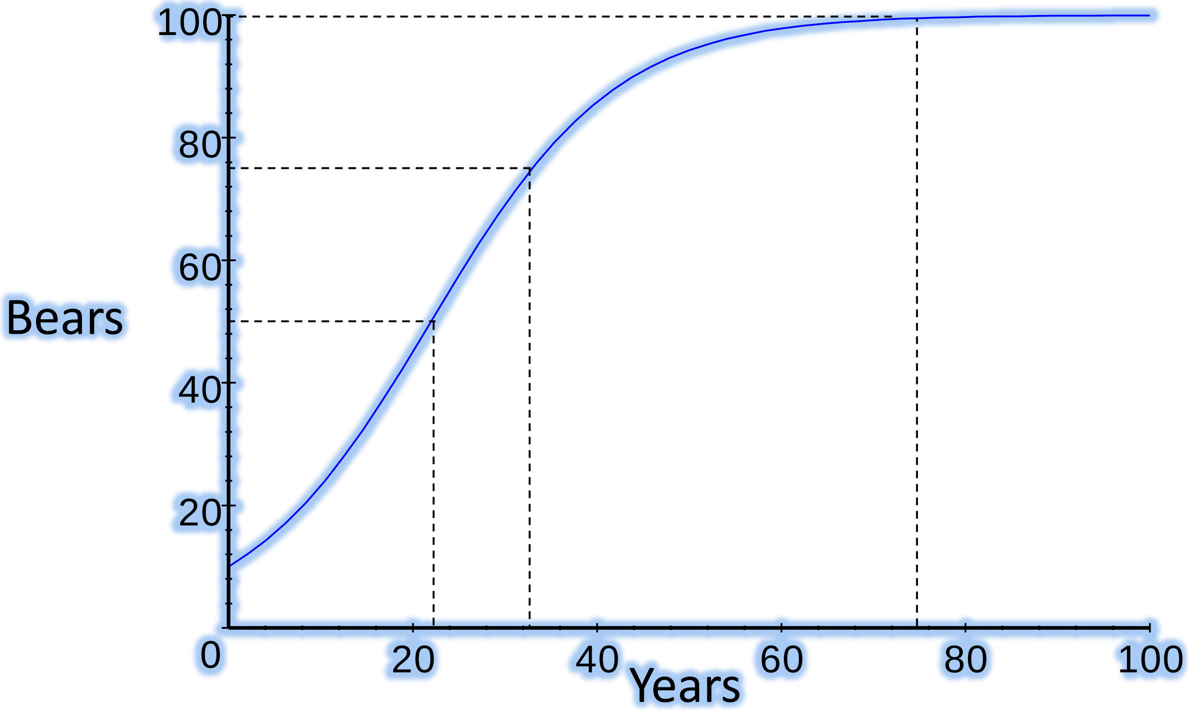
$$9e^{-10k} = \frac{77}{23}$$

$$e^{-10k} = 0.371981$$

$$\frac{-10k}{-10} = \frac{\ln 0.371981}{-10}$$

$$P = \frac{100}{1+9e^{-(0.09889)t}}$$

YOUR TURN



$$P = \frac{100}{1 + 9e^{-(0.09889)t}}$$

$Y = 50$ *at* 22 *years*

$Y = 75$ *at* 33 *years*

$Y = 100$ *at* 75 *years*

Example 5

In a particular town of 100,000 residents, 20,000 watched a viral video on the Internet. The rate of growth of the spread of information was jointly proportional to the amount of people who had not watched it. If 50% watched it after one hour, how long does 80% of the population watched the viral video?

$$L = 100,000$$

$$B = ?$$

$$(0, 20000)$$

$$(1, 50000)$$

$$P = \frac{L}{1 + Be^{-kt}}$$

$$B = \frac{100,000 - 20,000}{20,000}$$

$$B = 4$$

$$P(t) = \frac{100,000}{1 + 4e^{-kt}}$$

Example 5

In a particular town of 100,000 residents, 20,000 watched a viral video on the Internet. The rate of growth of the spread of information was jointly proportional to the amount of people who had not watched it. If 50% watched it after one hour, how long does 80% of the population watched the viral video?

$L = 100,000$

$B = ?$

$(0, 20000)$

$(1, 50000)$

$$P(t) = \frac{100,000}{1 + 4e^{-kt}}$$

$$50,000 = \frac{100,000}{1 + 4e^{-k(1)}}$$

$$50,000\left(1 + 4e^{k}\right) = 100,000$$

$$1 + 4e^{-k} = 2$$

Example 5

In a particular town of 100,000 residents, 20,000 watched a viral video on the Internet. The rate of growth of the spread of information was jointly proportional to the amount of people who had not watched it. If 50% watched it after one hour, how long does 80% of the population watched the viral video?

$$L = 100,000$$

$$B = ?$$

$$(0, 20000)$$

$$(1, 50000)$$

$$1 + 4e^{-k} = 2$$

$$4e^{-k} = 1$$

$$e^{-k} = \frac{1}{4}$$

$$\frac{1}{e^{k}} = \frac{1}{4}$$

$$e^{k} = 4$$

EXAMPLE 5

In a particular town of 100,000 residents, 20,000 watched a viral video on the Internet. The rate of growth of the spread of information was jointly proportional to the amount of people who had not watched it. If 50% watched it after one hour, how long does 80% of the population watched the viral video?

$$L = 100,000$$

$$B = ?$$

$$\left(0, 20000\right)$$

$$\left(1, 50000\right)$$

$$e^k = 4$$

$$k = \ln 4$$

$$P\left(t\right) = \frac{100,000}{1 + 4e^{-(\ln 4)t}}$$

$$80,000 = \frac{100,000}{1 + 4e^{-\ln 4t}}$$

Example 5

In a particular town of 100,000 residents, 20,000 watched a viral video on the Internet. The rate of growth of the spread of information was jointly proportional to the amount of people who had not watched it. If 50% watched it after one hour, how long does 80% of the population watched the viral video?

$$L = 100,000$$

$$B = ?$$

$$(0, 20000)$$

$$(1, 50000)$$

$$80,000 = \frac{100,000}{1 + 4e^{-\ln 4t}}$$

$$80,000\left(1 + 4e^{-\ln 4t}\right) = 100,000$$

$$1 + 4e^{-\ln 4t} = \frac{5}{4}$$

Example 5

In a particular town of 100,000 residents, 20,000 watched a viral video on the Internet. The rate of growth of the spread of information was jointly proportional to the amount of people who had not watched it. If 50% watched it after one hour, how long does 80% of the population watched the viral video?

$$L = 100,000$$
$$B = ?$$
$$(0, 20000)$$
$$(1, 50000)$$

$$1 + 4e^{-\ln 4t} = \frac{5}{4}$$

$$4e^{-\ln 4t} = \frac{1}{4}$$

$$e^{-\ln 4t} = \frac{1}{16}$$

$$e^{(-\ln 4)(t)} = \frac{1}{16}$$

$$-e^{\ln 4t} = \frac{1}{16}$$

$$-\ln e^{\ln 4t} = \ln \frac{1}{16}$$

$$\ln 4t = -\frac{\ln \frac{1}{16}}{\ln 4}$$

$$t = 2 \text{ hours}$$

AP Multiple Choice Practice Question 1 (non-calculator)

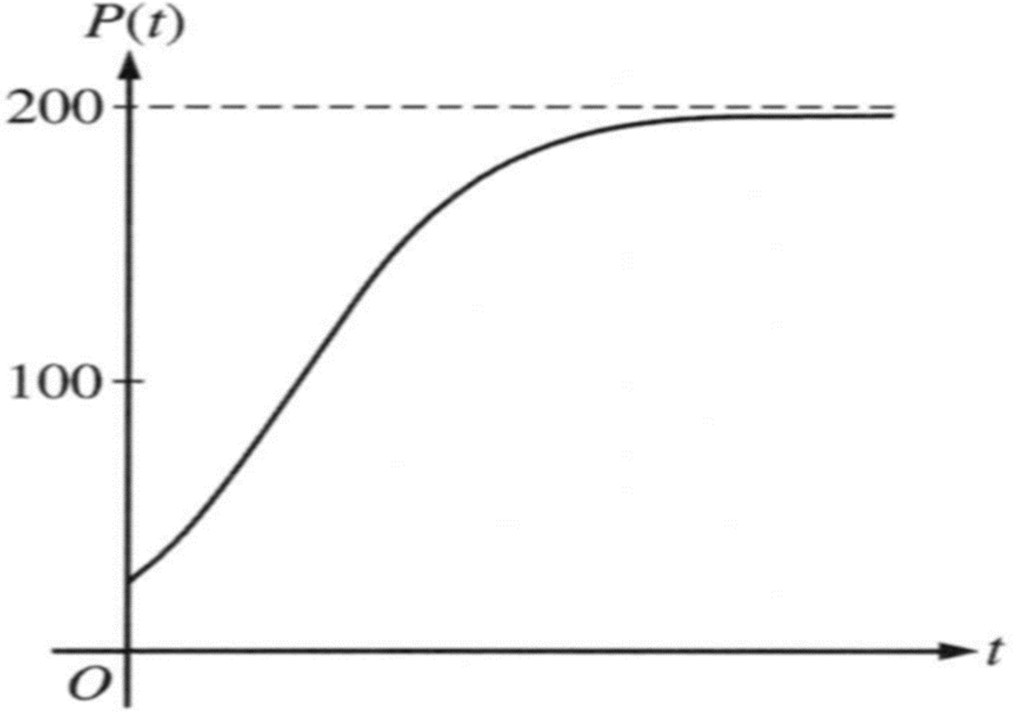
Which of the following differential equations for a population could model the logistic growth equation for the graph below?

(A) $\frac{dP}{dt} = 0.2P - 0.001P^2$

(B) $\frac{dP}{dt} = 0.1P - 0.001P^2$

(C) $\frac{dP}{dt} = 0.2P^2 - 0.001P$

(D) $\frac{dP}{dt} = 0.1P^2 - 0.001P$



AP Multiple Choice Practice Question 1 (non-calculator)

Which of the following differential equations for a population could model the logistic growth equation for the graph below?

Vocabulary	Connections and Process
Logistic Growth Equation (A) $\frac{dP}{dt} = 0.2P - 0.001P^2$ (B) $\frac{dP}{dt} = 0.1P - 0.001P^2$ (C) $\frac{dP}{dt} = 0.2P^2 - 0.001P$ (D) $\frac{dP}{dt} = 0.1P^2 - 0.001P$	$\frac{dP}{dt} = kP\left(1 - \frac{P}{200}\right); \frac{dP}{dt} = kP(200 - P)$ $\quad \frac{dP}{dt} = kP - \frac{kP^2}{200}$ A) $\frac{dP}{dt} = 0.2P - 0.001P^2$ $k = 0.2$ $\frac{dP}{dt} = 0.2P\left(1 - \frac{P}{200}\right) = 0.2P - \frac{1}{1000}P$ B) $\frac{dP}{dt} = 0.1P - 0.001P^2$ $k = 0.1$ $\frac{dP}{dt} = 0.1P\left(1 - \frac{P}{200}\right) \neq 0.1P - \frac{1}{2000}P$

AP Multiple Choice Practice Question 1 (non-calculator)

Which of the following differential equations for a population could model the logistic growth equation for the graph below?

Connections and Process	Answer
$\frac{dP}{dt} = kP\left(1 - \frac{P}{200}\right); \frac{dP}{dt} = kP(200 - P)$ $\quad \frac{dP}{dt} = kP - \frac{kP^2}{200}$ C) $\frac{dP}{dt} = 0.2P^2 - 0.001P$ $k = 0.001$ $\frac{dP}{dt} = 0.001P\left(1 - \frac{P}{200}\right) \neq 0.001P - \frac{1}{200000}P$ D) $\frac{dP}{dt} = 0.1P^2 - 0.001P$ $k = 0.001$ $\frac{dP}{dt} = 0.001P\left(1 - \frac{P}{200}\right) \neq 0.001P - \frac{1}{200000}P$	A

ASSIGNMENT

Worksheet