Tests for the Convergence and Divergence of Series

Test	Series	Conditions of Convergence	Conditions of Divergence	Comment
nth test for Divergence	$\sum_{n=1}^{\infty} a_n$	None	If $\lim_{n\to\infty} a_n \neq 0$ then series diverges	This test cannot be used to show convergence and the converse is not true.
Geometric series	$\sum_{n=0}^{\infty} ar^n$	$\|r\| < 1$	$\|r\| \geq 1$	Sum: $S = \frac{a}{1-r}$ where a is the 1st term
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+1})$	If in expanded form, the terms begin to "subtract out"	Not used for divergence	Infinite sum is the sum of the first few terms that do not subtract out.
P-Series	$\sum_{n=1}^{\infty} \frac{1}{n^p}$	$p > 1$	$p \leq 1$	If $p = 1$, then it is the divergent harmonic series
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$	Not used for divergence	Remainder: $\|R_N\| \leq \|a_{N+1}\|$
Integral Test (f is continuous, positive, and decreasing)	$\sum_{n=1}^{\infty} a_n$ where $a_n = f(n) \geq 0$	$\int_1^{\infty} f(x)dx$ converges	$\int_1^{\infty} f(x)dx$ diverges	Remainder: $0 < R_N < \int_N^{\infty} f(x)dx$
Ratio Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| < 1$	$\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| > 1$	Test is inconclusive if $\lim_{n\to\infty} \left\|\frac{a_{n+1}}{a_n}\right\| = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} < 1$	$\lim_{n\to\infty} \sqrt[n]{\|a_n\|} > 1$	Test is inconclusive if $\lim_{n\to\infty} \sqrt[n]{\|a_n\|} = 1$
Direct Comparison	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 < b_n \leq a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges	Note: $a_n > 0$ and $b_n > 0$
Limit Comparison	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $L > 0$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $L > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	Note: $a_n > 0$ and $b_n > 0$ and L must be finite

Borrowed heavily from Ron Larson's book Calculus, 8th ed. Published by Houghton Mifflin Company