Limits of Riemann's Sum

Section 4.2B

Calculus AP/Dual, Revised ©2018

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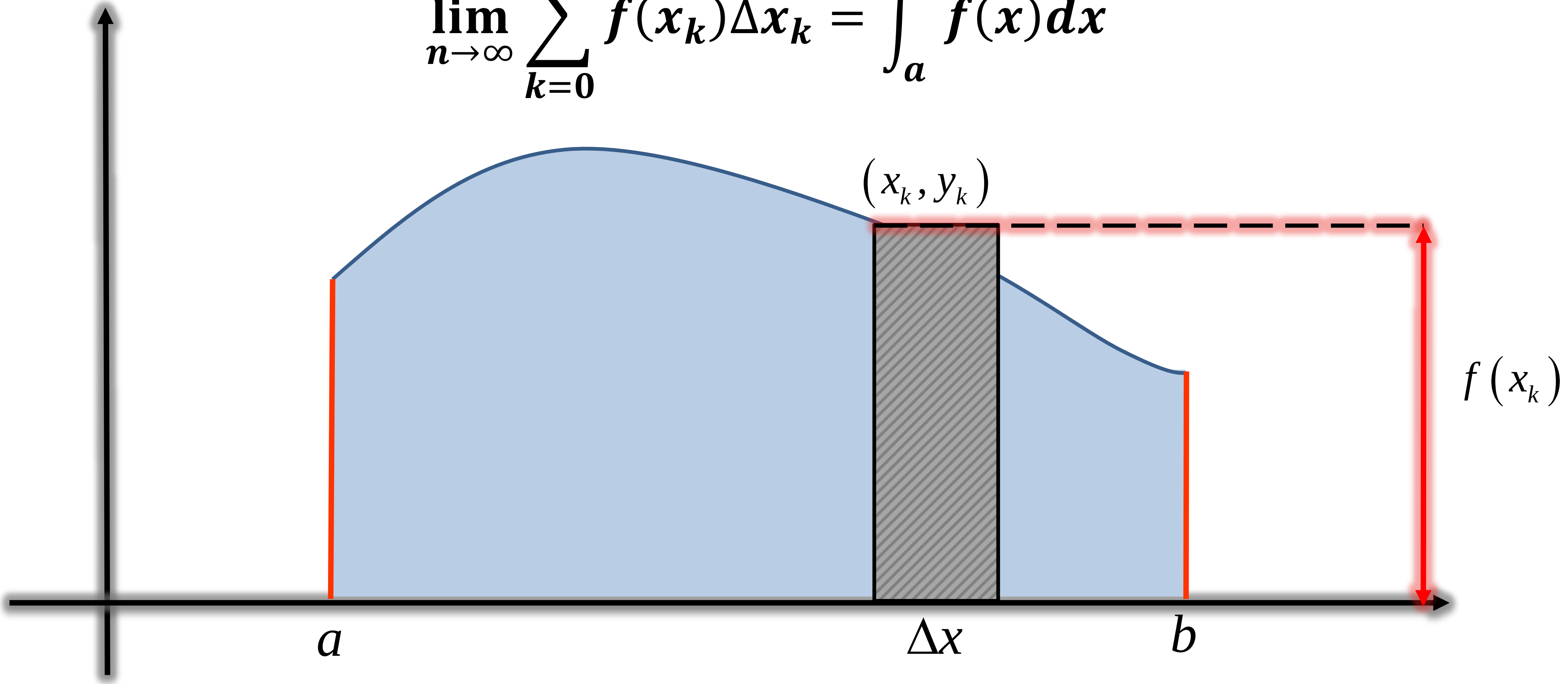
Summation Notation

$$\lim_{n\to\infty}\sum_{k=0}^{n} f(x_k)\Delta x_k = \int_a^b f(x)dx$$



Breaking Up Riemann's Sum

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k = \int_a^b f(x)\,dx$$

Height of the Rectangle

Width of the Rectangle = $\Delta x = \dfrac{(b-a)}{n}$

Review

$$\Delta x = \frac{b-a}{n}$$

$$x_K = a + (\Delta x)k$$

The Limit Equation for Riemann's Sum

Upper limit of summation: It tells us to end with $k = n$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

Index of summation

Lower limit of summation: It tells us to start with $k = 1$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\Delta x)\left[f\left(x_k\right)\right]$$

It is more traditional to use $k = 0$ for *left* or *midpoint* sums, and $k = 1$ for *right* sums.

Sigma Notation Breakdowns

A. $\sum_{k=1}^{n} 1 = n$

B. $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

C. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

D. $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$

E. $\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}, r \neq 1$

Example 1

Write the limit which is equal to $\int_0^5 x\,dx$ and then solve for the integral.

$$\lim_{n\to\infty}\sum_{k=1}^{n}(\Delta x)\left[f\left(x_k\right)\right]$$

$$a=0, b=5$$

$$\Delta x=\frac{b-a}{n}$$

$$\Delta x=\frac{5-0}{n}$$

$$f\left(x_k\right)=a+\left[(\Delta x)k\right]$$

Example 1

Write the limit which is equal to $\int_0^5 x\,dx$ and then solve for the integral.

$$x_k = a + (\Delta x)k$$

$$x_k = 0 + \left(\frac{5-0}{n}\right)k$$

$$f(x_k) = (x_k)$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}(\Delta x)\left[a + (\Delta x)k\right]$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{5-0}{n}\right)\left[\left(0 + \left(\frac{5-0}{n}\right)k\right)\right]$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{5}{n}\right)\left[\left(\frac{5}{n}k\right)\right]$$

Example 1

Write the limit which is equal to $\int_0^5 x\,dx$ and then solve for the integral.

$$\int_0^5 x\,dx$$

$$\left.\frac{x^2}{2}\right|_0^5 = \left(\frac{(5)^2}{2}\right) - \left(\frac{(0)^2}{2}\right)$$

$$\left(\frac{25}{2}\right) - \left(\frac{0}{2}\right)$$

$$\frac{25}{2} = 12.5$$

Example 1

Write the limit which is equal to $\int_0^5 x\,dx$ and then solve for the integral.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{5}{n}\right)\left[\left(\frac{5}{n}k\right)\right]$$

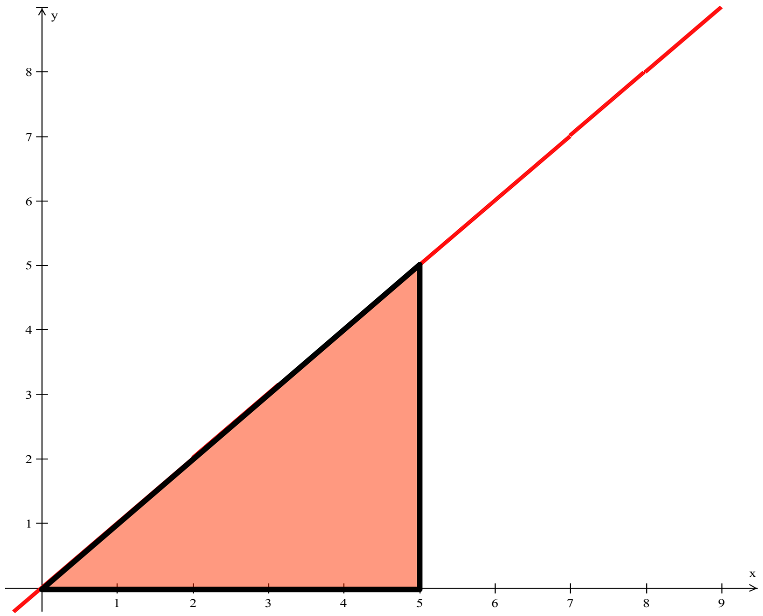
$$\frac{25}{2}=12.5$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{5}{n}\right)\left[\left(\frac{5}{n}k\right)\right]=\frac{25k}{n^2}\qquad \sum_{k=1}^{n}k=\frac{n(n+1)}{2}$$

$$\frac{25}{n^2}\sum_{k=1}^{n}k=\frac{25}{n^2}\frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}\frac{25}{n^2}\cdot\left(\frac{n}{2}\cdot\frac{n+1}{1}\right)$$

$$\lim_{n\to\infty}\frac{25}{2}\cdot\frac{n^2+n}{n^2}$$



Example 1

Write the limit which is equal to $\int_0^5 x\,dx$ and then solve for the integral.

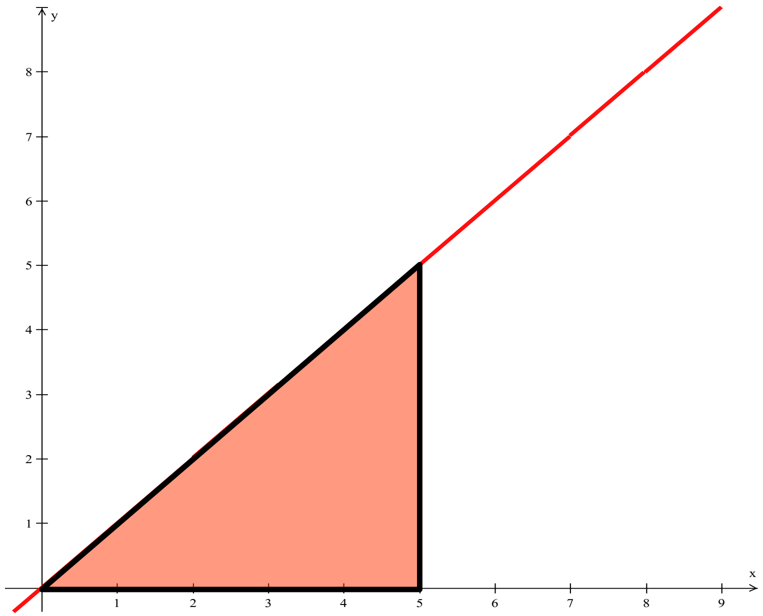
$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{5}{n}\right)\left[\left(\frac{5}{n}k\right)\right]$$

$$\frac{25}{2}=12.5$$

$$\lim_{n\to\infty}\frac{25}{2}\cdot\lim_{n\to\infty}\frac{n^2+n}{n^2}$$

$$\lim_{n\to\infty}\frac{25}{2}\cdot\left(\lim_{n\to\infty}\frac{n^2}{n^2}+\lim_{n\to\infty}\frac{1}{n^2}\right)$$

$$\lim_{n\to\infty}\frac{25}{2}\cdot(1+0)$$



$$\frac{25}{2}=12.5$$

Example 2

Write the limit which is equal to $\int_0^2 x^2\, dx$ and then solve for the integral.

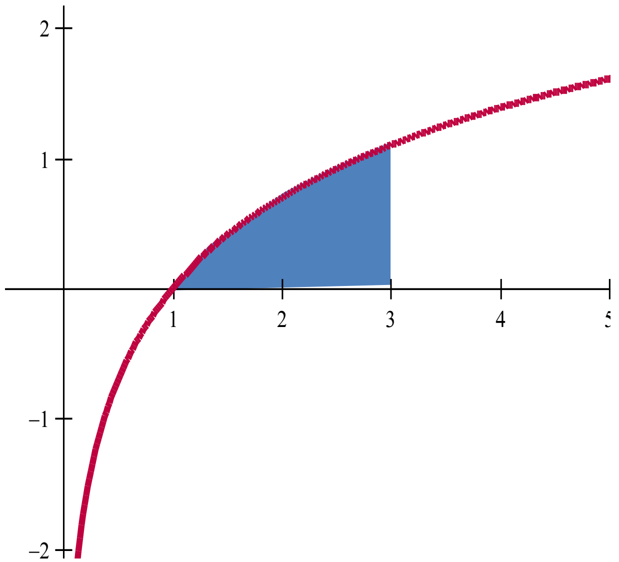
$$\lim_{n\to\infty}\sum_{k=1}^{n}(\Delta x)\left[a+(\Delta x)k\right]$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2-0}{n}\right)\left[\left(\left(\frac{2-0}{n}\right)k\right)^2\right]$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{n}\right)\left[\left(\frac{2}{n}k\right)^2\right]$$

EXAMPLE 3

Use the graph to write the limit for the area of the shaded region given the graph is $f(x) = \ln x$.



$$\lim_{n\to\infty}\sum_{k=1}^{n}(\Delta x)\left[f\left(x_k\right)\right]$$

$$a=1, b=3$$

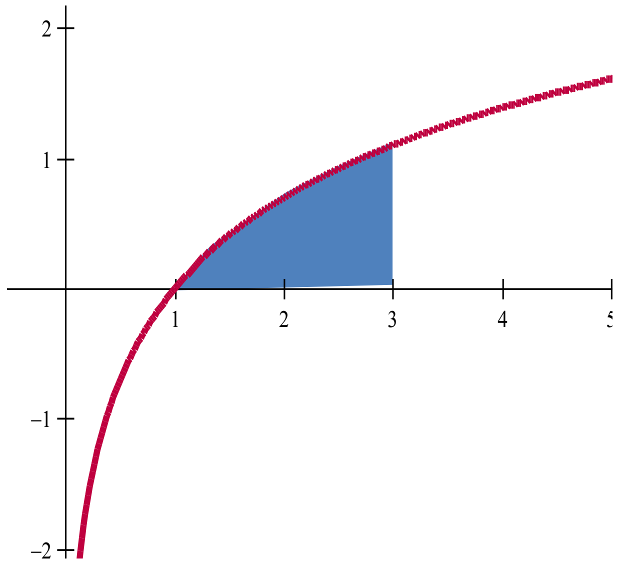
$$\Delta x=\frac{b-a}{n}$$

$$\Delta x=\frac{3-1}{n}$$

$$f\left(x_k\right)=a+\left(\Delta x\right)k$$

EXAMPLE 3

Use the graph to write the limit for the area of the shaded region given the graph is $f(x) = \ln x$.



$$x_k = a + (\Delta x)k$$

$$x_k = 1 + \left(\frac{3-1}{n}\right)k$$

$$f(x_k) = \ln x$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{n}\right)\left[\ln\left(1+\frac{2}{n}k\right)\right]$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(\left(1+\frac{2}{n}k\right)\right)\left(\frac{2}{n}\right)$$

YOUR TURN

Write the limit which is equal to $\int_2^5 (3x^2 + 1)\, dx$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3}{n}\right)\left[3\left(2+\frac{3}{n}k\right)^2+1\right]$$

Example 4

Write the limit using Integral Notation, $\lim\limits_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3}{n}\right) \sqrt[3]{\frac{3k}{n}}$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\Delta x)\left[a + (\Delta x)k\right]$$

$$a = 0$$

$$b = 3$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{b-a}{n}\right)\left[a + (\Delta x)k\right]$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{b-a}{n}\right)\left[\sqrt[3]{(0) + (\Delta x)k}\right]$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{(3)-(0)}{n}\right)\sqrt[3]{\left[(0) + \left(\frac{(3)-(0)}{n}\right)k\right]}$$

EXAMPLE 4

Write the limit using Integral Notation, $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{3}{n}\right)\sqrt[3]{\frac{3k}{n}}$.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{(3)-(0)}{n}\right)\sqrt[3]{\left[(0)+\left(\frac{(3)-(0)}{n}\right)k\right]}$$

$a = 0$

$b = 3$

$$f(x)=\sqrt[3]{\left[(0)+\left(\frac{(3)-(0)}{n}\right)k\right]}$$

$$f(x)=\sqrt[3]{x}$$

$$\int_0^3 \sqrt[3]{x}\,dx$$

Example 5

Write the limit using Integral Notation,

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{n}\right)\left[\left(3+\frac{2k}{n}\right)^3+\left(3+\frac{2k}{n}\right)\right].$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}(\Delta x)\left[a+(\Delta x)k\right]$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)\left[\left(a+(\Delta x)k\right)^3+\left(a+(\Delta x)k\right)\right]$$

$a=3$

$b=??$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{b-(3)}{n}\right)\left[\left((3)+(\Delta x)k\right)^3+\left((3)+(\Delta x)k\right)\right]$$

$b-(3)=2$

$b=5$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{(5)-(3)}{n}\right)\left[\left((3)+\left(\frac{(5)-(3)}{n}\right)k\right)^3+\left((3)+\left(\frac{(5)-(3)}{n}\right)k\right)\right]$$

EXAMPLE 5

Write the limit using Integral Notation,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2}{n}\right)\left[\left(3+\frac{2k}{n}\right)^3+\left(3+\frac{2k}{n}\right)\right].$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{(5)-(3)}{n}\right)\left[\left((3)+\left(\frac{(5)-(3)}{n}\right)k\right)^3+\left((3)+\left(\frac{(5)-(3)}{n}\right)k\right)\right]$$

$$f(x)=\left[\left(x\right)^3+\left(x\right)\right]$$

$$\int_3^5 \left(x^3+x\right)\,dx$$

EXAMPLE 6

Write the limit using Integral Notation, $\lim_{n\to\infty} \sum_{k=1}^{n} \left[1 + \left(1 + \frac{7k}{n}\right)^5\right]\left(\frac{7}{n}\right)$.

$$\int_1^8 \left(1 + x^5\right)\, dx$$

YOUR TURN

Write the limit using Integral Notation, $\lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{3}{n}\right)\left[\left(2+\frac{3k}{n}\right)^4+4\right]$.

$$\int_2^5 \left(x^4+4\right)\,dx$$

LEFT AND RIGHT ENDPOINTS

A. **Left-Side Endpoint equation:** $\lim_{n\to\infty} \sum_{k=0}^{n-1} f(C_K)\Delta x_K$

B. **Right-Side Endpoint equation:** $\lim_{n\to\infty} \sum_{k=1}^{n} f(C_K)\Delta x_K$

Example 7

Write a left-endpoint and right-endpoint Riemann Sum R_4 that approximates $\int_0^1 \sqrt{x}\ dx$. Do not solve.

$$\Delta x = \frac{b-a}{n}$$

$$\Delta x = \frac{(1)-(0)}{4} = \frac{1}{4}$$

$$x = a + (\Delta x)k$$

$$x = (0) + \left(\frac{1}{4}\right)k$$

Example 7a

Write a left-endpoint and right-endpoint Riemann Sum R_4 that approximates $\int_0^1 \sqrt{x}\ dx$. Do not solve.

$$x = (0) + \left(\frac{1}{4}\right)k$$

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f\left(C_K\right)\left(\Delta x\right)$$

Intervals:

$$\left(0, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{3}{4}, 1\right)$$

$$\lim_{n\to\infty} \sum_{k=0}^{4-1} f\left(C_K\right)\left(\Delta x\right)$$

$$\lim_{n\to\infty} \sum_{k=0}^{3} \left(\frac{1}{4}\right)\left[\sqrt{\left(0+\frac{1(0)}{4}\right)} + \sqrt{\left(0+\frac{1(1)}{4}\right)} + \sqrt{\left(0+\frac{1(2)}{4}\right)} + \sqrt{\left(0+\frac{1(3)}{4}\right)}\right]$$

$$LR_4 = \left(\frac{1}{4}\right)\left[\sqrt{0} + \sqrt{\left(\frac{1}{4}\right)} + \sqrt{\left(\frac{2}{4}\right)} + \sqrt{\left(\frac{3}{4}\right)}\right]$$

Example 7b

Write a left-endpoint and right-endpoint Riemann Sum R_4 that approximates $\int_0^1 \sqrt{x}\ dx$. Do not solve.

$$x = (0) + \left(\frac{1}{4}\right)k$$

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(C_K)(\Delta x)$$

Intervals:

$$\left(0, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{4}\right), \left(\frac{3}{4}, 1\right)$$

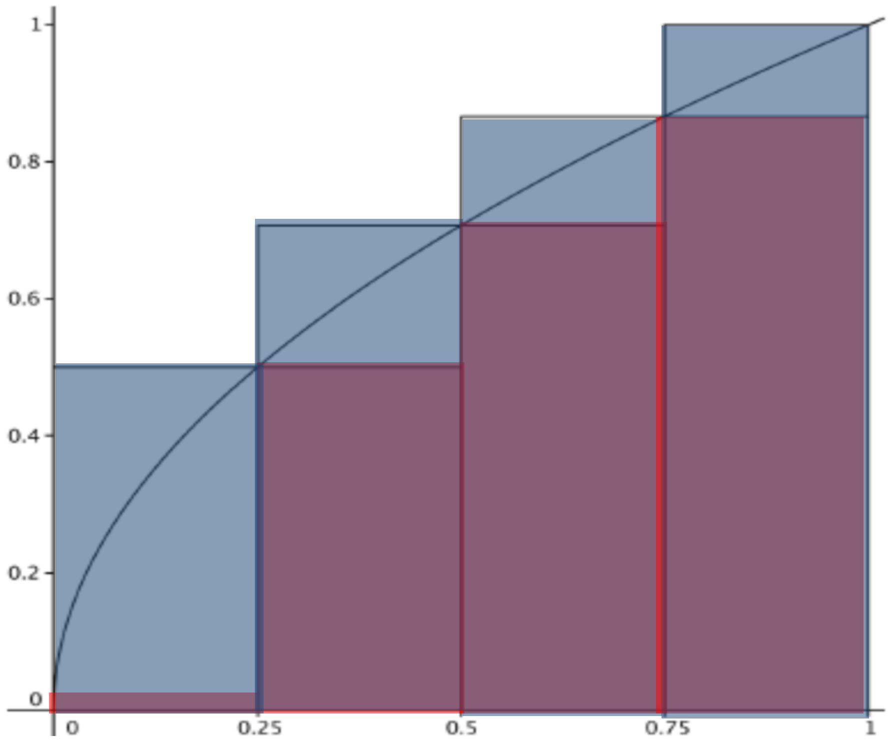
$$\lim_{n\to\infty}\sum_{k=1}^{4} f(C_K)(\Delta x)$$

$$\lim_{n\to\infty}\sum_{k=1}^{4}\left(\frac{1}{4}\right)\left[\sqrt{\left(0+\frac{1(1)}{4}\right)}+\sqrt{\left(0+\frac{1(2)}{4}\right)}+\sqrt{\left(0+\frac{1(3)}{4}\right)}+\sqrt{\left(0+\frac{1(4)}{4}\right)}\right]$$

$$RR_4 = \left(\frac{1}{4}\right)\left[\sqrt{\left(\frac{1}{4}\right)}+\sqrt{\left(\frac{2}{4}\right)}+\sqrt{\left(\frac{3}{4}\right)}+\sqrt{\left(\frac{4}{4}\right)}\right]$$

EXAMPLE 7

Write a left-endpoint and right-endpoint Riemann Sum R_4 that approximates $\int_0^1 \sqrt{x}\ dx$. Do not solve.



$$LR_4 = \left(\frac{1}{4}\right)\left[\sqrt{0} + \sqrt{\left(\frac{1}{4}\right)} + \sqrt{\left(\frac{2}{4}\right)} + \sqrt{\left(\frac{3}{4}\right)}\right]$$

$$RR_4 = \left(\frac{1}{4}\right)\left[\sqrt{\left(\frac{1}{4}\right)} + \sqrt{\left(\frac{2}{4}\right)} + \sqrt{\left(\frac{3}{4}\right)} + \sqrt{\left(\frac{4}{4}\right)}\right]$$

YOUR TURN

Write a left-endpoint and right-endpoint Riemann Sum R_{50} that approximates $\int_0^1 \sqrt{x}\ dx$. Do not solve.

$$LR_{50} = \left(\frac{1}{50}\right)\left[\sqrt{0} + \sqrt{\frac{1}{50}} + \sqrt{\frac{2}{50}} + \sqrt{\frac{3}{50}} + \ldots + \sqrt{\frac{49}{50}}\right]$$

$$RR_{50} = \left(\frac{1}{50}\right)\left[\sqrt{\frac{1}{50}} + \sqrt{\frac{2}{50}} + \sqrt{\frac{3}{50}} + \ldots + \sqrt{\frac{50}{50}}\right]$$

AP Multiple Choice Practice Question 1 (non-calculator)

Which of the following approximates the area between $g(x) = \frac{5}{x} + 2$ and the x-axis on the interval $[1, 7]$ using a Right Riemann sum with 9 equal subdivisions?

(A) $\sum_{k=1}^{9} \frac{15}{3+2k} + 2$

(B) $\sum_{k=0}^{8} \frac{15}{3+2k} + 2$

(C) $\sum_{k=1}^{9} \left(\frac{15}{3+2k} + 2\right) \cdot \frac{2}{3}$

(D) $\sum_{k=0}^{8} \left(\frac{15}{3+2k} + 2\right) \cdot \frac{2}{3}$

AP Multiple Choice Practice Question 1
(non-calculator)

Which of the following approximates the area between $g(x)=\frac{5}{x}+2$ and the x-axis on the interval $[1,7]$ using a Right Riemann sum with 9 equal subdivisions?

Vocabulary	Process and Connections	Answer
Right Riemann's Sum 9 Equal Subintervals	$\sum_{k=1}^{n}(\Delta x)\, f(x_K)$ $RightRS, k=1$ $\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{2}{3}\right)\left(\frac{5}{\left(\frac{3+2k}{3}\right)}+2\right)$ $\Delta x=\frac{7-(1)}{9}=\frac{6}{9}=\frac{2}{3}$ $x_K=a+(\Delta x)k$ $x_K=1+\left(\frac{2}{3}\right)k=\frac{3+2k}{3}$ $\lim_{n\to\infty}\sum_{k=1}^{9}\left(\frac{2}{3}\right)\left(\frac{15}{3+2k}+2\right)$	C

ASSIGNMENT

Worksheet