Second Derivative Test

Section 3.4

Calculus AP/Dual, Revised ©2017

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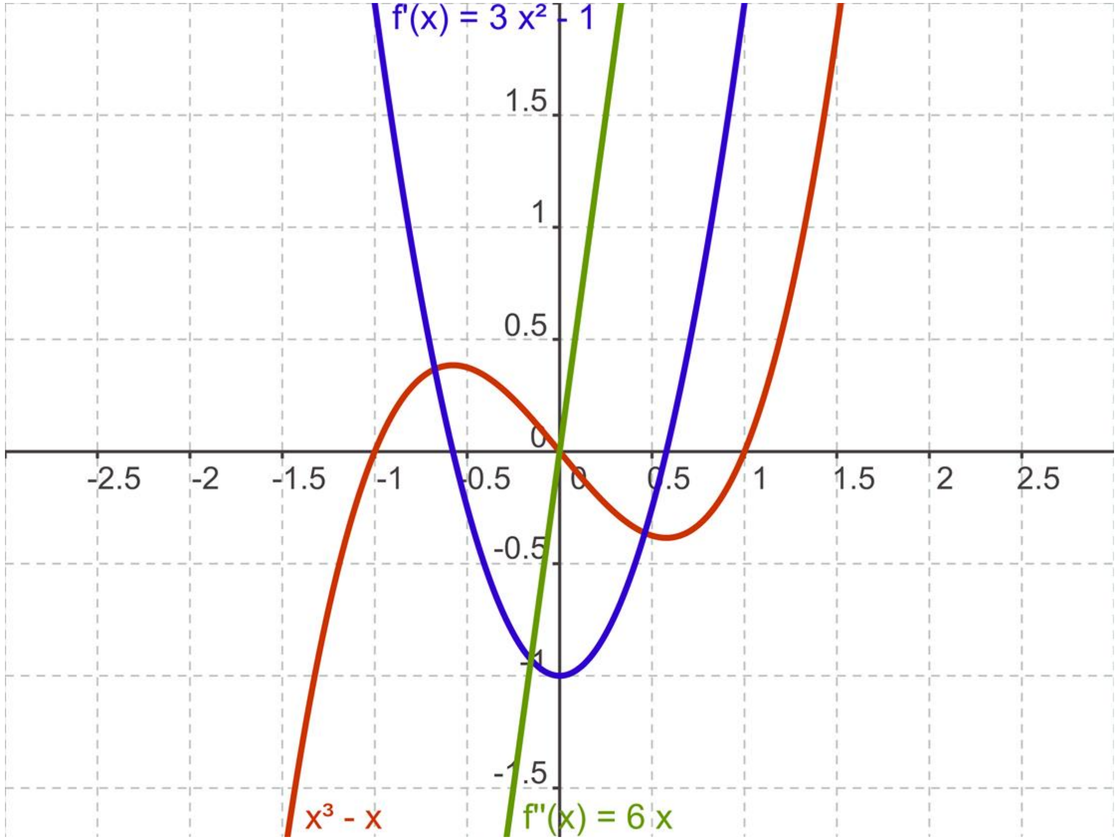
Fill in the Blank Review

A. First Derivative Test and Concavity

1. **f' positive then f is: ___INCREASING___**
2. **f' negative then f is: ___DECREASING___**
3. **f'' positive then f is Concave _UP__**
4. **f'' negative then f is Concave _DOWN_**
5. **$f'(c) = 0$ or $f'(c) = DNE$ to which c is x of a critical point, then the point is called __CRITICAL POINT__**
6. **$f''(x) = 0$ or $f''(x) = DNE$ and $f''(x)$ changes signs, then the point is called _POINT OF INFLECTION_**

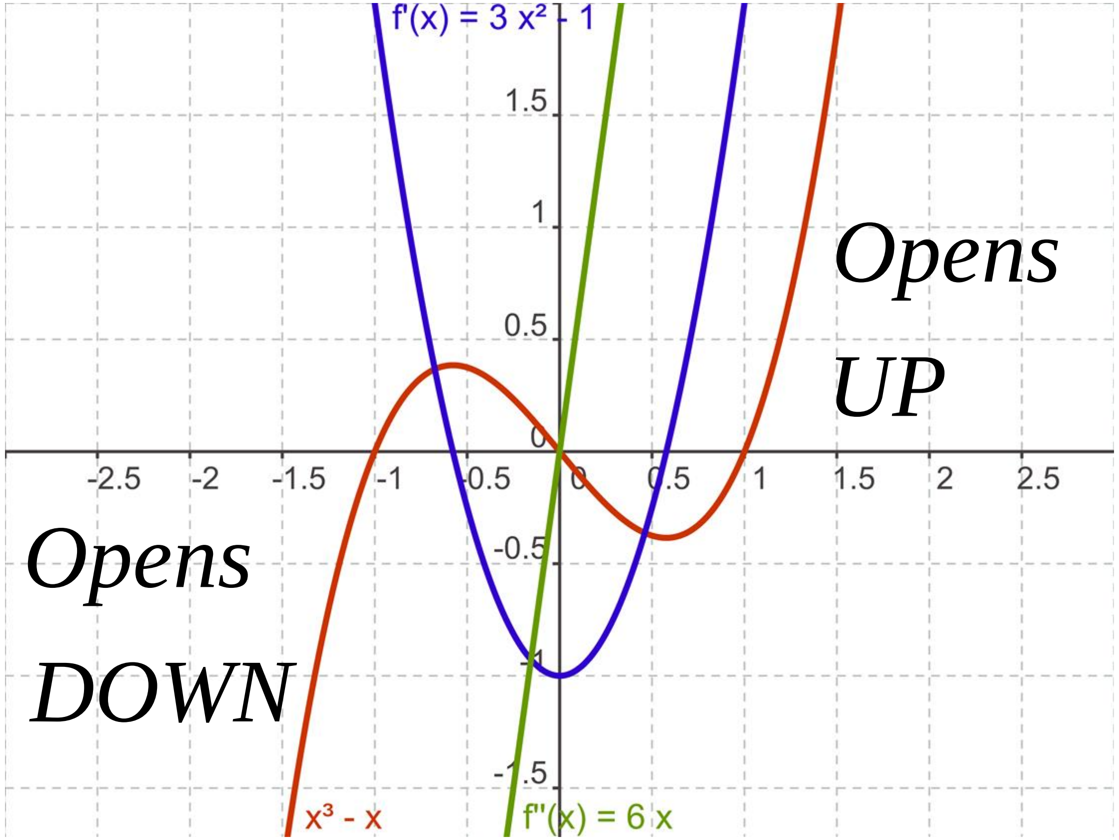
Review Example

Graph $f(x) = x^3 - x$, $f'(x)$, and $f''(x)$



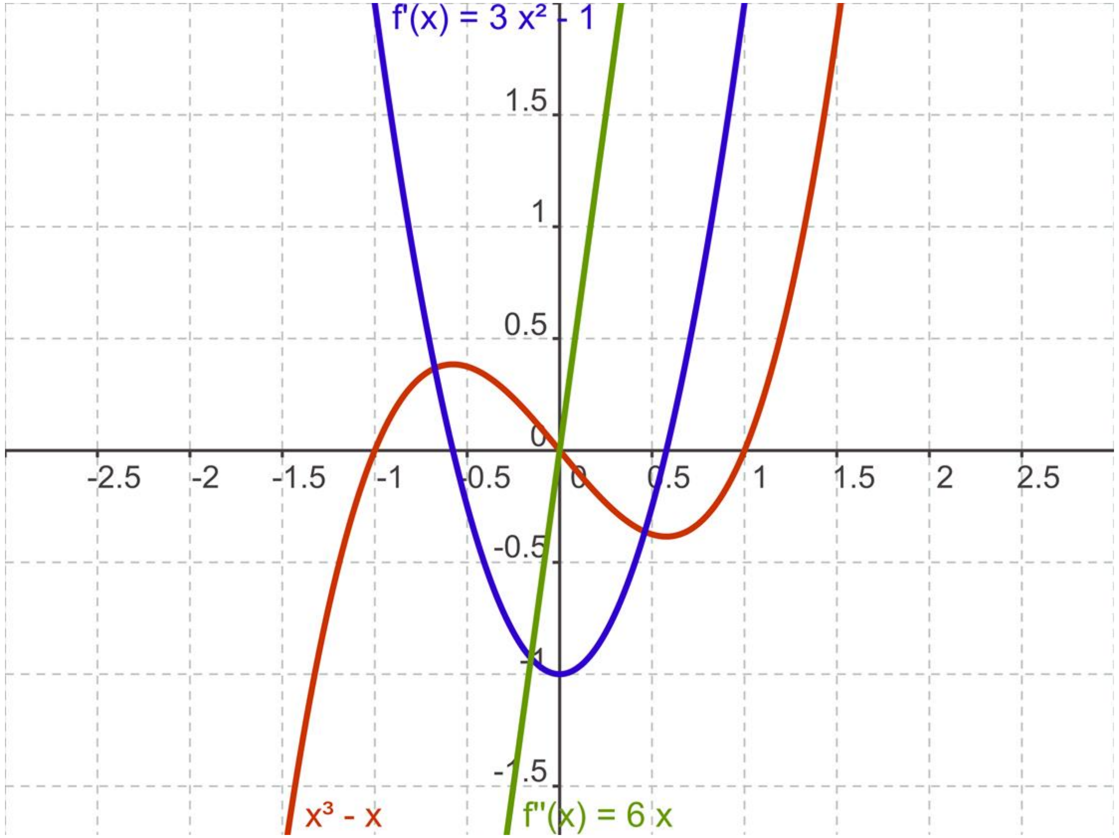
Review Example

Compare $f(x) = x^3 - x$ and $f''(x)$



Review Example

Solve for $f'(c) = 0$ of $f(x) = x^3 - x$ and then take $f''(c)$



$$f(x) = x^3 - x$$

$$f'(x) = 3x^2 - 1$$

$$3x^2 - 1 = 0$$

$$3x^2 = 1$$

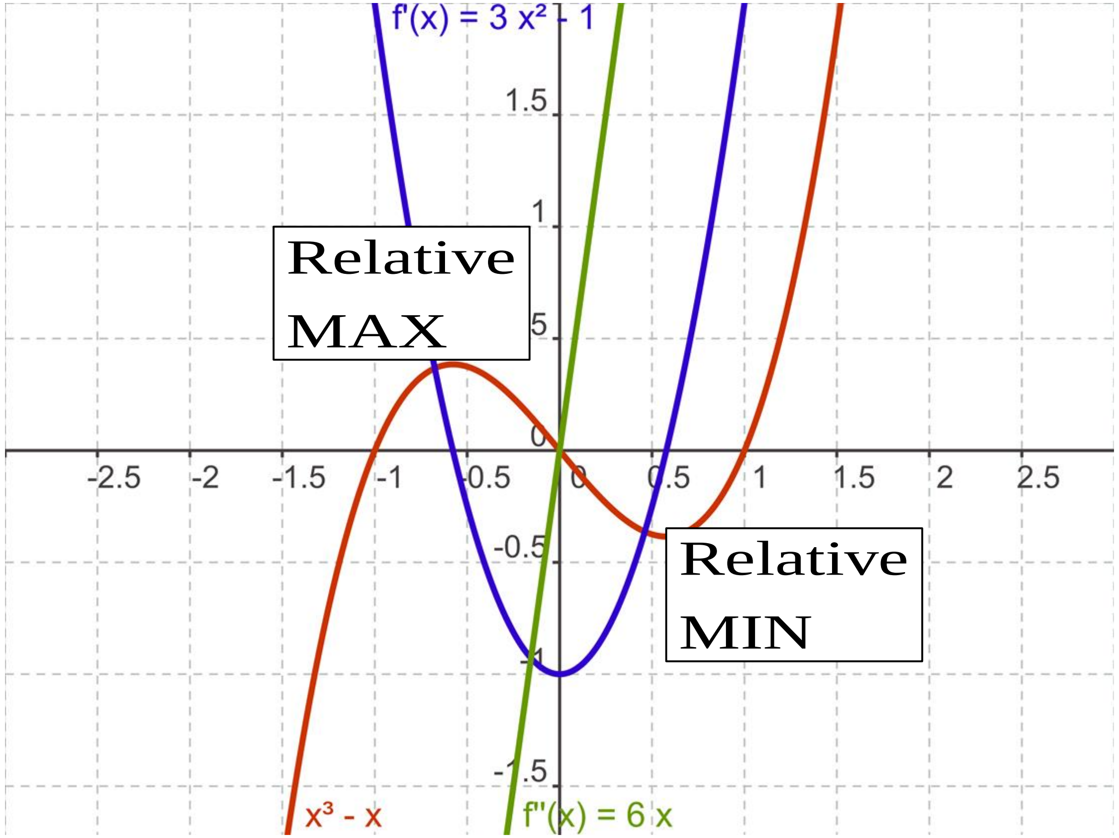
$$x^2 = \frac{1}{3}$$

$$x = \pm\frac{1}{\sqrt{3}}$$

$$C.N.$$

Review Example

Solve for $f'(c) = 0$ of $f(x) = x^3 - x$ and then take $f''(c)$



$$x = \pm \frac{1}{\sqrt{3}}$$

$$C.N.$$

$$f'(x) = 3x^2 - 1$$

$$f''(x) = 6x$$

$$f''\left(-\frac{1}{\sqrt{3}}\right) = -\frac{6}{\sqrt{3}}$$

$$f''\left(\frac{1}{\sqrt{3}}\right) = +\frac{6}{\sqrt{3}}$$

F’ vs F’’

f ’ Test	f ’’ Test for Concavity
Critical Points	Concavity
Absolute/Global Extrema	Point of Inflection
Relative/Local Max or Min	Process f ’’ = 0 or DNE

Second Derivative Test

A. Let f be a function such that $f'(c) = 0$ and f'' exists on an open interval containing c:

1. If $f''(c) > 0$, then $f(x)$ has a <u>relative minimum</u> at $(c, f(c))$ and f is concave up
2. If $f\,''(c) < 0$, then $f(x)$ has a <u>relative maximum</u> at $(c, f(c))$ and f is concave down
3. If $f'(x) = 0$ and $f''(x) = 0$, then $f(x)$ is <u>inconclusive</u>. Use the first derivative test.

STEPS

A. Identify all critical points using the first derivative.
B. Determine the second derivative
C. Plug in the critical points into the second derivative
   1. If the critical point is positive, there is a relative minimum
   2. If the critical point is negative, there is a relative maximum
   3. If the critical point is zero, test is inconclusive and test must revert back to the first derivative test

EXAMPLE 1

Using the Second Derivative Test to find the relative extrema of

$$f(x) = x + \frac{2}{x}.$$

$$f\left(x\right) = x + 2x^{-1}$$

$$f'\left(x\right) = 1 - 2x^{-2}$$

$$f'\left(x\right) = 1 - \frac{2}{x^2}$$

TAKE CRITICAL POINT

$$CP: 1 - \frac{2}{x^2} = 0$$

take common denominator

$$\frac{x^2}{x^2} - \frac{2}{x^2} = 0$$

$$x^2 - 2 = 0, x^2 = 0$$

$$x = 0, \pm\sqrt{2}$$

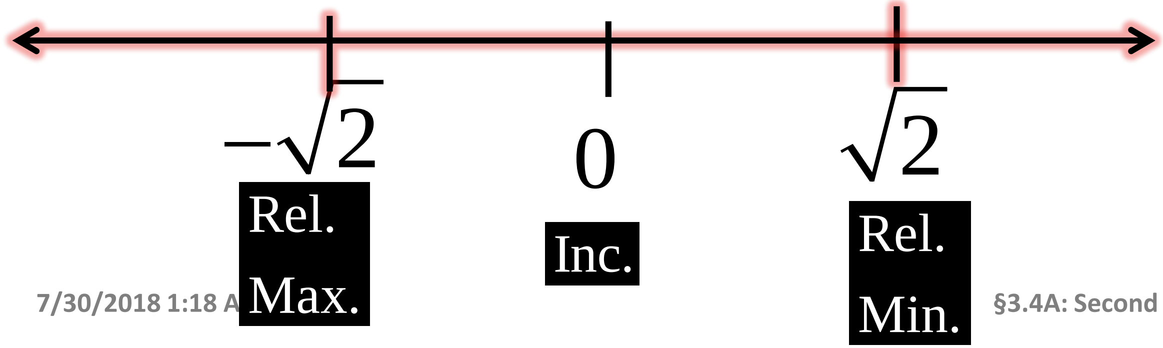
$$CP: x = 0, \pm\sqrt{2}$$

Example 1

Using the Second Derivative Test to find the relative extrema of $f(x) = x + \frac{2}{x}$ and justify.

$$f'(x) = 1 - \frac{2}{x^2}$$

$$f''(x) = 4x^{-3} = \frac{4}{x^3}$$



$f''\left(-\sqrt{2}\right) = -$ *Concave Down*

$f''(0) =$ *Inc.*   *Inc.*

$f''\left(\sqrt{2}\right) = +$ *Concave Up*

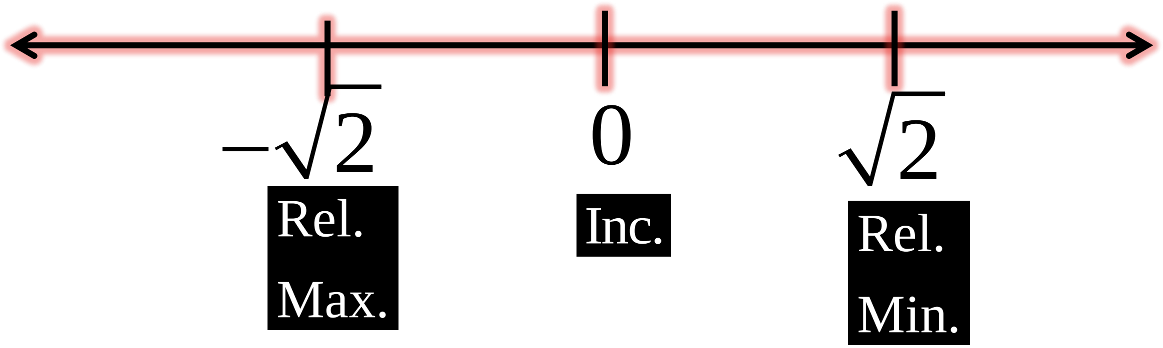
$$f''\left(-\sqrt{2}\right) = \frac{4}{\left(-\sqrt{2}\right)^3} = \frac{4}{-2\sqrt{2}} = NEG(-)$$

$$f''(0) = \frac{4}{(0)^3} = Und$$

$$f''\left(\sqrt{2}\right) = \frac{4}{\left(\sqrt{2}\right)^3} = \frac{4}{2\sqrt{2}} = POS(+)$$

EXAMPLE 1

Using the Second Derivative Test to find the relative extrema of $f(x) = x + \frac{2}{x}$ and justify.



$f(x)$ has a Relative Minimum at $x = \sqrt{2}$ when $f'' > 0$

$f(x)$ has a Relative Maximum at $x = -\sqrt{2}$ when $f'' < 0$

When $x = 0$, the test is inconclusive and must use the first derivative test to determine relative extrema.

Example 1

Using the Second Derivative Test to find the relative extrema of $f(x) = x + \frac{2}{x}$ and justify.

$f(x)$ has a Relative Minimum at $x = \sqrt{2}$ when $f'' > 0$

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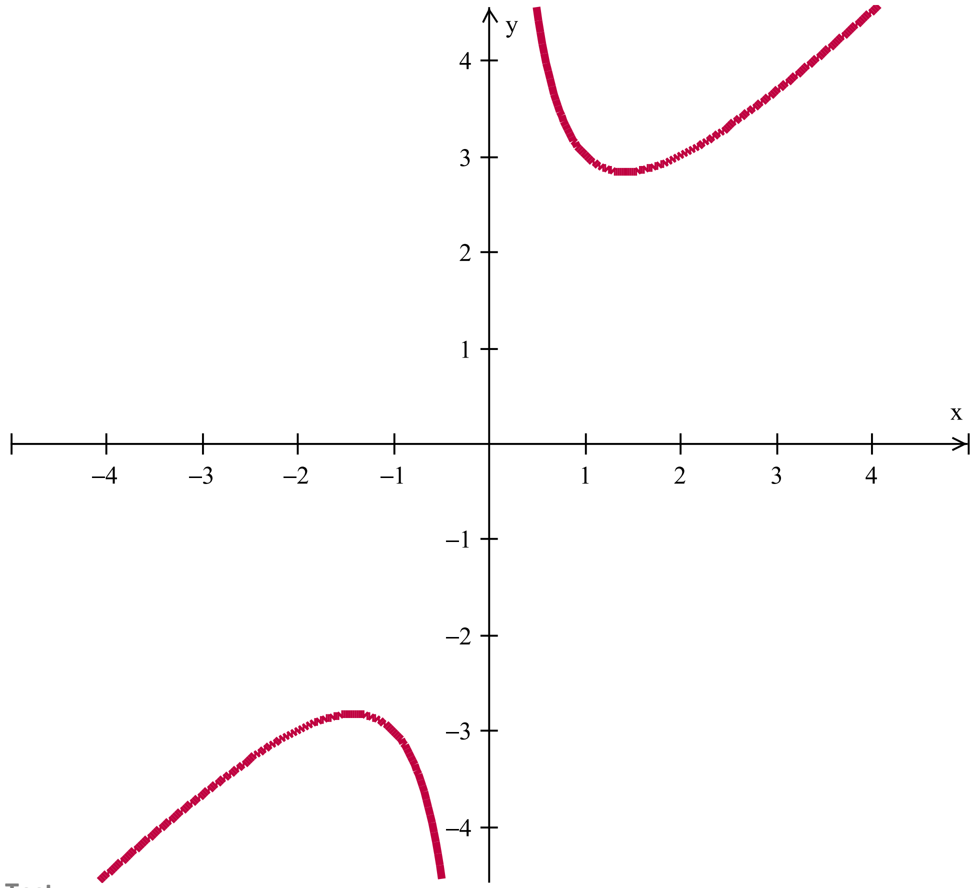
When $x = 0$, the test is inclusive and must use the first derivative test to determine relative extrema.



EXAMPLE 2

Using the Second Derivative Test to find the relative extrema of $f(x) = -3x^5 + 5x^3$ and justify.

$$f(x) = -3x^5 + 5x^3$$

$$f'(x) = -15x^4 + 15x^2$$

$$CP: -15x^2(x^2 - 1) = 0$$

$$CP: x = 0, x = \pm 1$$

EXAMPLE 2

Using the Second Derivative Test to find the relative extrema of $f(x) = -3x^5 + 5x^3$ and justify.

$$f'(x) = -15x^4 + 15x^2$$

$$f''(x) = -60x^3 + 30x$$

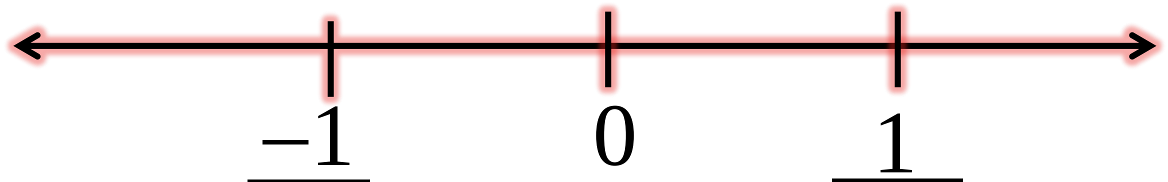
$$f''(0) = 0$$

$$f''(1) = -30$$

$$f''(-1) = 30$$

Example 2

Using the Second Derivative Test to find the relative extrema of $f(x) = -3x^5 + 5x^3$ and justify.



Rel. Min.

If $f''(0)=0$, then we revert back to the First Derivative Test to decide the Relative Maxima.

Rel. Max.

$$f''(0) = 0$$

$$f''(1) = -30$$

$$f''(-1) = 30$$

$$f(-1) = -3(-1)^5 + 5(-1)^3$$

$$f(-1) = 2$$

$$f(1) = -3(1)^5 + 5(1)^3$$

$$f(1) = -2$$

$f(x)$ has a Relative Minimum at $(-1,-2)$ when $f'' > 0$

$f(x)$ has a Relative Maximum at $(1,2)$ when $f'' < 0$

At $f''(0) = 0$ it is inconclusive and must use the first derivative test to determine relative extrema.

Example 2

Using the Second Derivative Test to find the relative extrema of $f(x) = -3x^5 + 5x^3$ and justify.

$f(x)$ has a Relative Minimum at $(-1,-2)$ when $f'' > 0$

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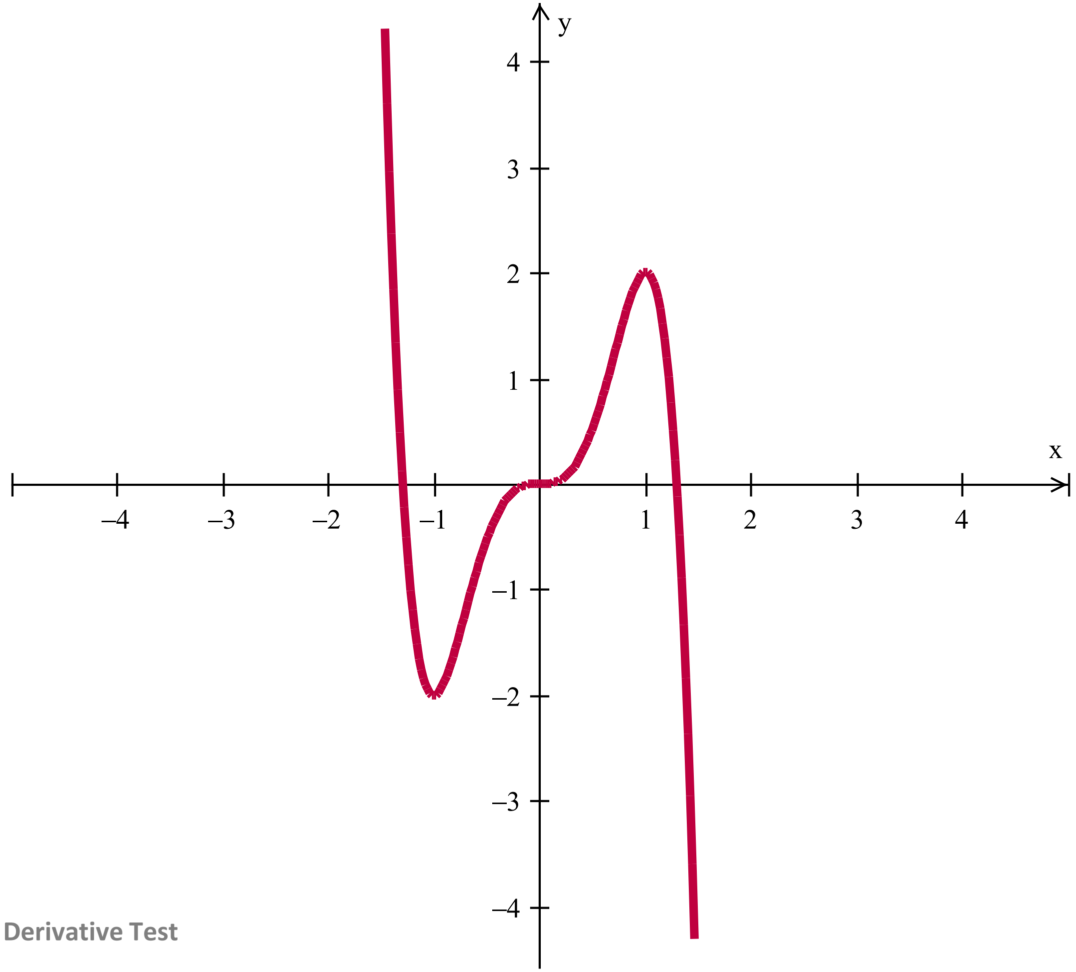
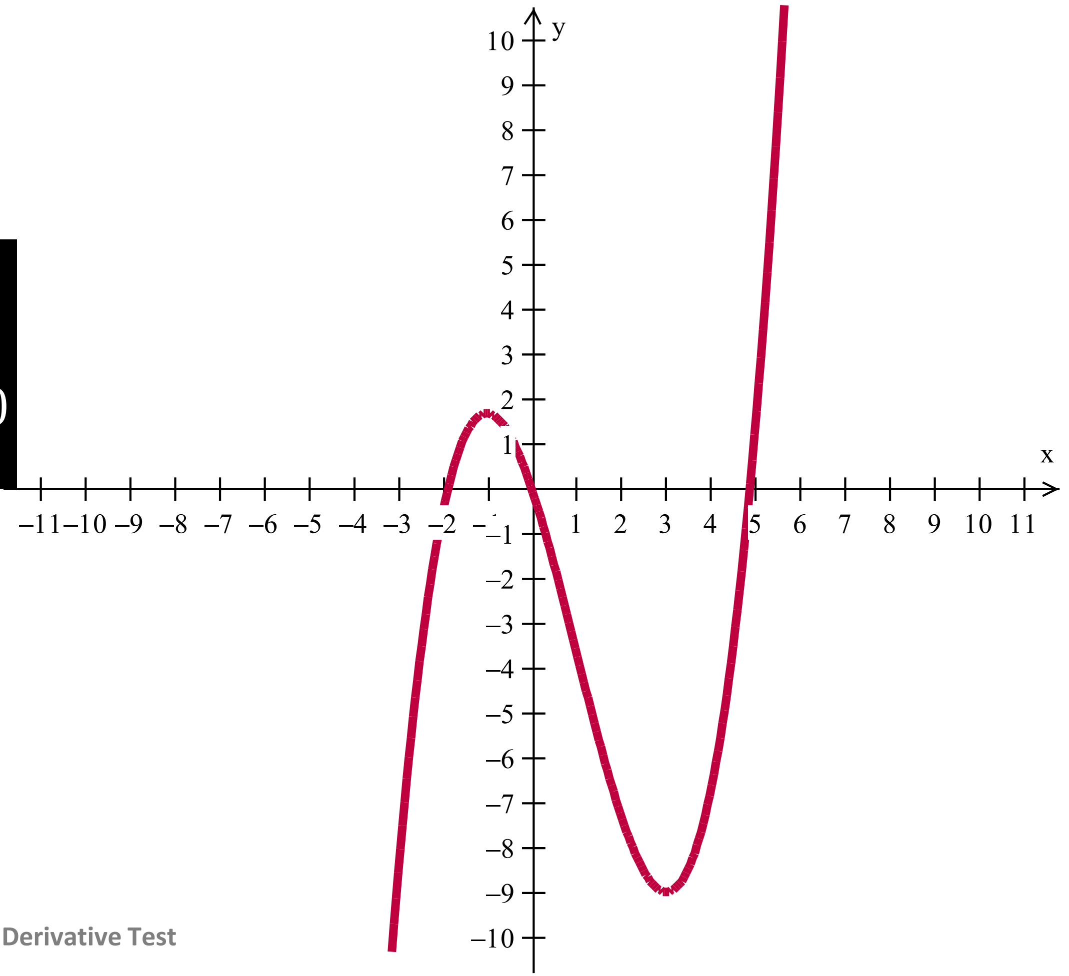


YOUR TURN

Using the Second Derivative Test to find the relative extrema of $f(x) = \frac{1}{3}x^3 - x^2 - 3x$ and justify.

$f(x)$ has a Relative Minimum at $(3, -9)$ when $f'' > 0$

$f(x)$ has a Relative Maximum at $\left(-1, \frac{5}{3}\right)$ when $f'' < 0$



EXAMPLE 3

Suppose that the function f has a continuous second derivative for all x and that $f(-1) = 2$, $f'(-1) = -3$, $f''(-1) = 5$. Let g be a function whose derivative is given by $g'(x) = \left(x^4 - 6x^3\right)\left(3f(x) + 2\,f'(x)\right)$ for all x. Write an equation of the tangent line to the graph of f at the point of where $x = -1$. Justify response.

$$f\left(-1\right) = 2,$$

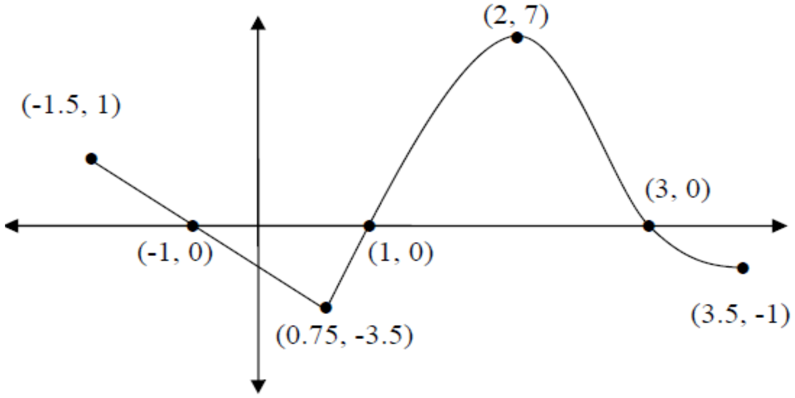
$$f\,'\left(-1\right) = -3$$

$$\left(-1, 2\right)$$

$$y - 2 = -3\left(x + 1\right)$$

EXAMPLE 4

The figure below shows the graph of the derivative of f, f' on the closed interval $[-1.5, 3.5]$. The graph of f' has a horizontal tangent line at $x = 2$ and is linear on the interval $[-1.5, 0.75]$.



Graph of f'

(a) Find the x-coordinates of the relative maxima of f. Justify your answer.

(b) Find the x-coordinates of the points of inflection of f. Justify your answer.

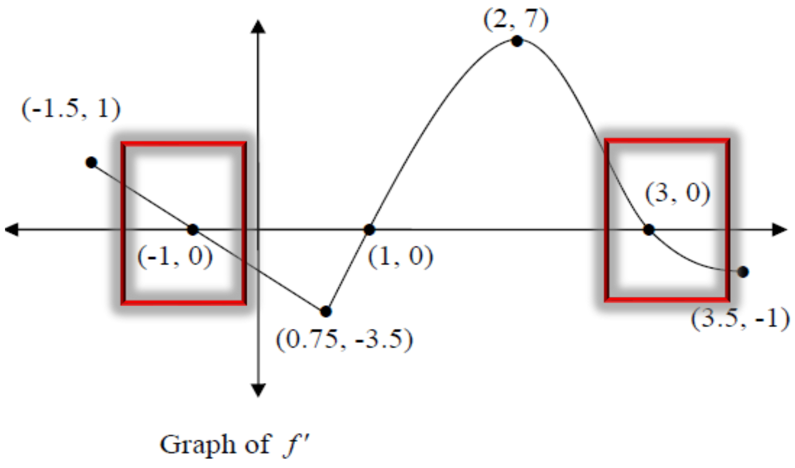
(c) On what intervals is f decreasing? Justify your answer.

(d) Is the function f twice-differentiable? Justify your answer.

EXAMPLE 4A

The figure below shows the graph of the derivative of f, f' on the closed interval $[-1.5, 3.5]$. The graph of f' has a horizontal tangent line at $x = 2$ and is linear on the interval $[-1.5, 0.75]$.

(a) Find the x-coordinates of the relative maxima of f. Justify your answer.



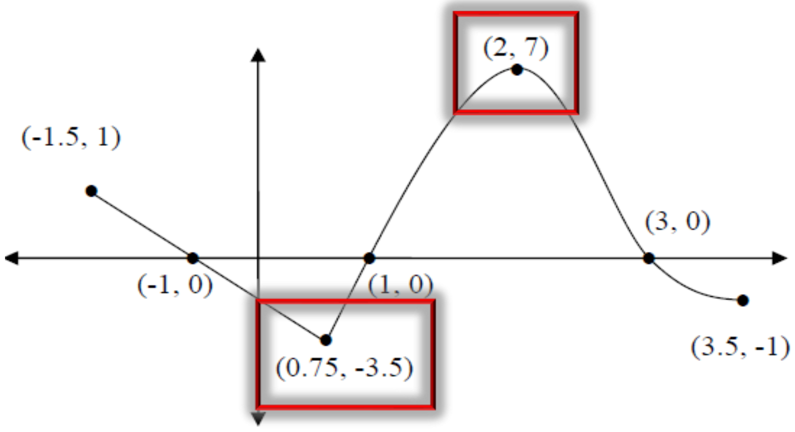
$f(x)$ has a relative maximum at $x = -1, x = 3$

where f' changes sign from positive to negative

EXAMPLE 4B

The figure below shows the graph of the derivative of f, f' on the closed interval $[-1.5, 3.5]$. The graph of f' has a horizontal tangent line at $x = 2$ and is linear on the interval $[-1.5, 0.75]$.

(b) Find the x-coordinates of the points of inflection of f. Justify your answer.



Graph of f'

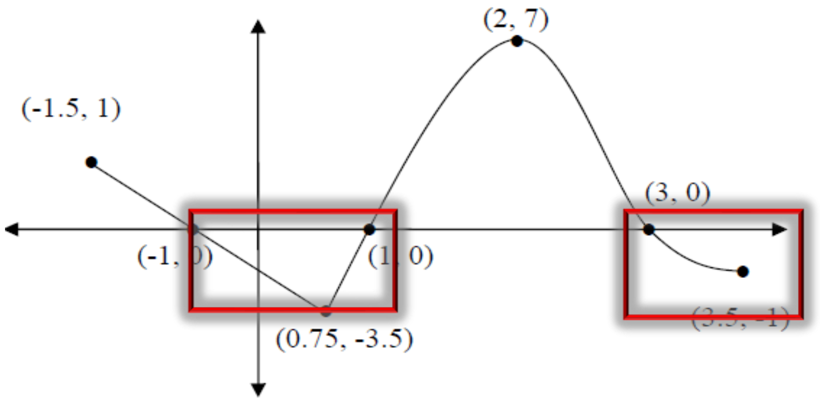
f has a POI at $(0.75, -3.5)$ where f' changes from decreasing to increasing.

f has a POI at $(2, 7)$ where f' changes from increasing to decreasing.

EXAMPLE 4C

The figure below shows the graph of the derivative of f, f' on the closed interval $[-1.5, 3.5]$. The graph of f' has a horizontal tangent line at $x = 2$ and is linear on the interval $[-1.5, 0.75]$.

(c) On what intervals is f decreasing? Justify your answer.



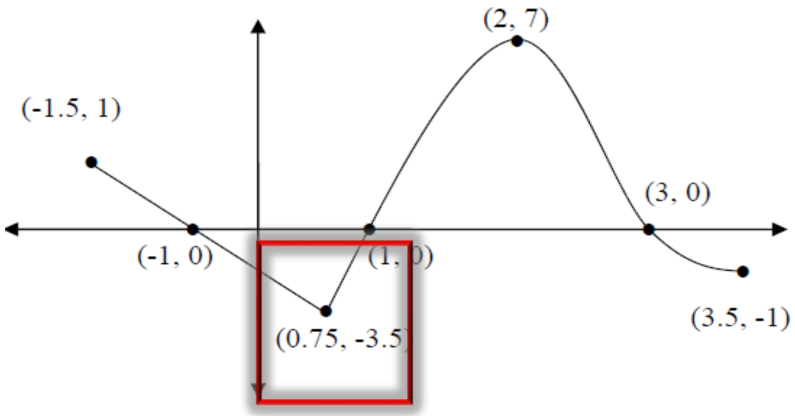
Graph of f'

f is decreasing where $f' < 0$ at $I(-1,1) \cup I(3,3.5)$.

EXAMPLE 4D

The figure below shows the graph of the derivative of f, f' on the closed interval $[-1.5, 3.5]$. The graph of f' has a horizontal tangent line at $x = 2$ and is linear on the interval $[-1.5, 0.75]$.

(d) Is the function f twice-differentiable (Continuous and differentiable)? Justify your answer.



Graph of f'

f is continuous and not differentiable because $x = 0.75$ has a sharp turn and therefore, not differentiable.

To Recap

A. The First Derivative tells us:
   1. Extrema
   2. Relative Maximum
   3. Relative Minimum
B. The Second Derivative tells us:
   1. Concavity
   2. Points of Inflection
C. The Second Derivative TEST tells us:
   1. Extrema

AP Multiple Choice Practice Question 1 (non-calculator)

Let h be twice differentiable. $h'(x) > 0$ and $h''(x) > 0$ for all reals. What is a possible value for $h(3)$ if $h(0) = 0$, $h(1) = 2$ and $h(2) = 7$?

(A) 12

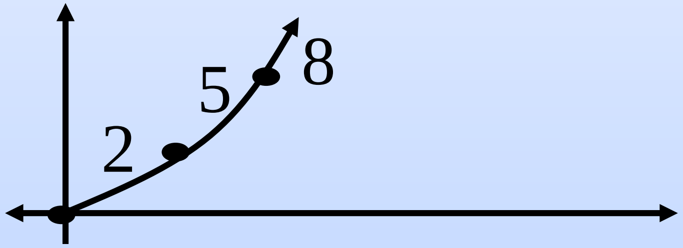
(B) 7

(C) 4

(D) 14

AP Multiple Choice Practice Question 1 (Non-Calculator)

Let h be twice differentiable. $h'(x) > 0$ and $h''(x) > 0$ for all reals. What is a possible value for $h(3)$ if $h(0) = 0$, $h(1) = 2$ and $h(2) = 7$?

Vocabulary	Connections and Process	Answer and Justifications
1st Derivative Test 2nd Derivative Test	$h'(x) > 0:$ increasing $\quad 7+8=15$ $h''(x) > 0:$ concave up $\quad$ close to 14 $(0,0),(1,2),(2,7)$ 	**D** As $h'(x) > 0$ and $h''(x) > 0,$ the function is increasing and concave up. The only possible answer is 14.

Assignment

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31-40 all & Justify all responses