Implicit Differentiation

Section 2.5

Calculus AP/Dual, Revised ©2017

viet.dang@humbleisd.net

REVIEW

Solve y' or $\frac{d}{dx}$ of $x^4 + 3y = 6x$

$$x^4 + 3y = 6x$$

$$3y = -x^4 + 6x$$

$$y = -\frac{x^4}{3} + 2x$$

$$y' = 2 - \frac{4}{3}x^3$$

Review

$$y' = 2 - \frac{4}{3}x^3$$

This form is written in <u>explicit</u> form, as it means it represents what y equals (or where the variable is on one side)

Review

Solve $\frac{d}{dx}$ of $7x^2$

$$\frac{d}{dx}\left[7x^2\right]$$

$$7\frac{d}{dx}\left[x^2\right]$$

$$7\left(2x\right)\frac{dx}{dx}$$

$$14x$$

General Practice

Take the derivative in respects to x or known as $\frac{d}{dx}$

1) x	$1\frac{dx}{dx}=1(1)=1$
2) $2x^3$	$6x^2\frac{dx}{dx}=6x^2$
3) $4x^5$	$20x^4\frac{dx}{dx}=20x^4$
4) $2\cos(x)$	$-2\sin x\frac{dx}{dx}=-2\sin x$

General Practice

Take the derivative in respects to y or known as $\frac{dy}{dx}$

1) y	$1\frac{dy}{dx} = 1(1) = 1$
2) $2y^3$	$6y^2\frac{dy}{dx}$
3) $4y^5$	$20y^4\frac{dy}{dx}$
4) $2\cos(y)$	$-2\sin y\frac{dy}{dx}$

DEFINITIONS

A. **Implicit Differentiation takes the derivative with both sides with respects to x in a relationship between x, y, and $\frac{dy}{dx}$; essentially an application of the CHAIN RULE**

B. **An IMPLICIT function is written to which one has to complete additional steps to isolate the variable. Functions written are not solved for *y* as it is not clearly stated but it is implied.**

C. **Examples include:**

   1. $y = \frac{x}{2}$ **whereas it can be written implicitly as** $xy = 1$
   2. $x^2 - 2y^3 + 4y = 2$ **can be written with implicit differentiation**

STEPS

A. Differentiate both sides of the equation with respect to x. MUST Include a $\frac{dy}{dx}$ or y' every time to differentiate with y

B. Collect all $\frac{dy}{dx}$ on LEFT side and all other "stuff" to the right side

C. Factor out $\frac{dy}{dx}$

D. Divide both sides by the extra "stuff"

Example 1

Solve $\frac{dy}{dx}$ of $x^2 + y^2 = 25$

$$x^2 + y^2 = 25$$

$$\frac{d}{dx}\left[x^2\right] + \frac{d}{dx}\left[y^2\right] = \frac{d}{dx}[25]$$

Do we have any y terms or any terms we took the derivative in respects to y?

$$2x + 2y = 0$$

$$\frac{d}{dy}2y\frac{dy}{dx}$$

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

EXAMPLE 1

Solve $\frac{dy}{dx}$ of $x^2 + y^2 = 25$

$$2x\frac{dx}{dx} + 2y\frac{dy}{dx} = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{2y\frac{dy}{dx}}{2y} = \frac{-2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 2

Solve $\frac{dy}{dx}$ of $2x^3 = 2y^2 + 5$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

EXAMPLE 3

Solve $\frac{dy}{dx}$ of $3y^3 + y^2 - 2y + 4x^2 = 7$

$$\frac{dy}{dx} = \frac{-8x}{9y^2 + 2y - 2}$$

Your Turn

Solve $\frac{dy}{dx}$ of $y^3 + y^2 - 5y - x^2 = -4$

$$\frac{dy}{dx} = \frac{2x}{3y^2 + 2y - 5}$$

EXAMPLE 4

Solve $\frac{dy}{dx}$ of $x^3y^3 - y = x$

$$x^3y^3 - y = x$$

x^3y^3 is being multiplied: Use the PRODUCT RULE

$$\frac{d}{dx}\left(x^3y^3\right) - \frac{d}{dx}(y) = \frac{d}{dx}(x)$$

$$\left(x^3 3y^2\right)\frac{dy}{dx} + y^3 3x^2 + (-1)\frac{dy}{dx} = 1$$

$$\left(x^3 3y^2\right)\frac{dy}{dx} + (-1)\frac{dy}{dx} = 1 - 3x^2y^3$$

Example 4

Solve $\frac{dy}{dx}$ of $x^3y^3 - y = x$

$$\left(x^3 3y^2\right)\frac{dy}{dx} + (-1)\frac{dy}{dx} = 1 - 3x^2y^3$$

$$\frac{dy}{dx}\left(3x^3y^2 - 1\right) = 1 - 3x^2y^3$$

$$\frac{dy}{dx} = \frac{1 - 3x^2y^3}{3x^3y^2 - 1}$$

Example 5

Solve $\frac{dy}{dx}$ of $\sin y \cos x = 5x$

$$\frac{dy}{dx} = \frac{5 + \sin y \sin x}{\cos y \cos x}$$

EXAMPLE 6

Solve $\frac{dy}{dx}$ of $\sin x = e^y$ in $(0, \pi)$ in terms of x.

$$\sin x = e^y$$

$$\cos x \frac{dx}{dx} = e^y \frac{dy}{dx}$$

$$\frac{\cos x}{e^y} = \frac{e^y \frac{dy}{dx}}{e^y}$$

$$\frac{dy}{dx} = \frac{\cos x}{e^y}$$

Example 6

Solve $\frac{dy}{dx}$ of $\sin x = e^y$ in $(0, \pi)$ in terms of x.

$$\frac{dy}{dx} = \frac{\cos x}{e^y}$$

$$\frac{dy}{dx} = \frac{\cos x}{\sin x}$$

$$\boxed{\frac{dy}{dx} = \cot x}$$

YOUR TURN

Solve $\frac{dy}{dx}$ of $2\sin x \cos y = 1$

$$\frac{dy}{dx} = \frac{\cos x \cos y}{\sin x \sin y}$$

Example 7

Solve the second derivative of $x^2 + y^2 = 16$

$$x^2 + y^2 = 16$$

$$2x + 2y = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Example 7

Solve the second derivative of $x^2 + y^2 = 16$

$$\frac{dy}{dx} = \frac{-x}{y}$$

$$\frac{d^2y}{dx^2} = \frac{y\left(-1\right) - \left(-x\right)\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y + x\frac{dy}{dx}}{y^2}$$

Example 7

Solve the second derivative of $x^2 + y^2 = 16$

$$\frac{d^2y}{dx^2} = \frac{-y + x\left(\dfrac{-x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y - \dfrac{x^2}{y}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\dfrac{y^2}{y} - \dfrac{x^2}{y}}{y^2}$$

EXAMPLE 7

Solve the second derivative of $x^2 + y^2 = 16$

$$\frac{d^2y}{dx^2} = \frac{-\frac{y^2}{y} - \frac{x^2}{y}}{y^2}$$

$$\frac{d^2y}{dx^2} = \frac{-y^2 - x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-\left(x^2 + y^2\right)}{y^3}$$

Example 7

Solve the second derivative of $x^2 + y^2 = 16$

$$\frac{d^2y}{dx^2} = \frac{-\left(x^2 + y^2\right)}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-(16)}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-16}{y^3}$$

Example 8

Solve the equation of the tangent line to the graph of $x^2 + y^2 = 25$ at the point $(3, -4)$.

$$x^2 + y^2 = 25$$

$$2x + 2y = 0$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

EXAMPLE 8

Solve the equation of the tangent line to the graph of $x^2 + y^2 = 25$ at the point $(3, -4)$.

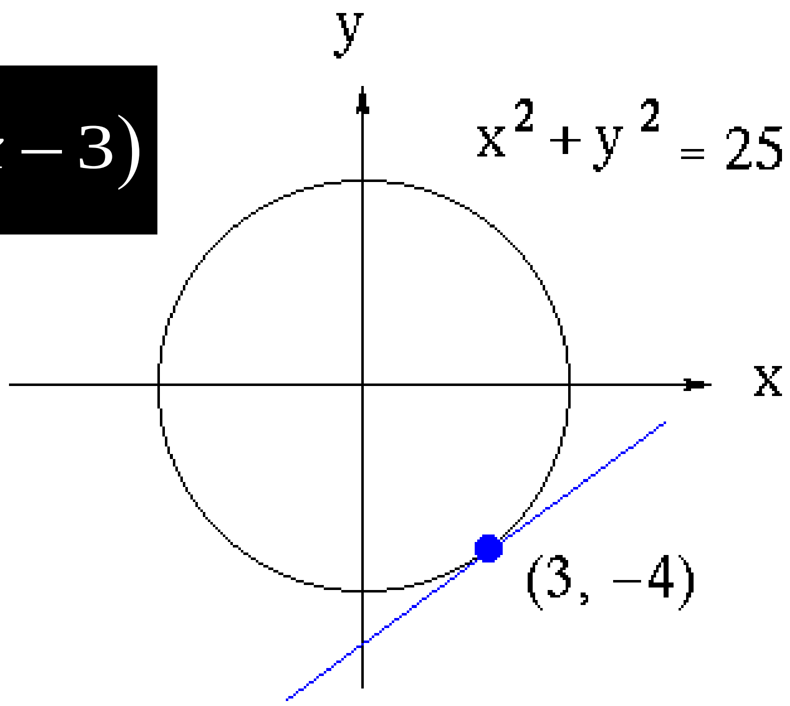
$$\frac{dy}{dx} = \frac{3}{4}$$

$$y + 4 = \frac{3}{4}(x - 3)$$

Example 8

Solve the equation of the tangent line to the graph of $x^2 + y^2 = 25$ at the point $(3, -4)$.

$$y + 4 = \frac{3}{4}(x - 3)$$



Example 9

Find the equation of the tangent line to the graph of $7x^2 + xy + 7y^2 = 15$ at the point $(1, 1)$.

$$7x^2 + xy + 7y^2 = 15$$

$$14x + x\frac{dy}{dx} + y + 14y\frac{dy}{dx} = 0$$

$$x\frac{dy}{dx} + 14y\frac{dy}{dx} = -14x - y$$

$$\frac{dy}{dx}(x + 14y) = -14x - y$$

EXAMPLE 9

Find the equation of the tangent line to the graph of $7x^2 + xy + 7y^2 = 15$ at the point $(1, 1)$.

$$\frac{dy}{dx}(x+14y) = -14x - y$$

$$\frac{dy}{dx} = \frac{-14x - y}{x+14y}$$

$$\frac{dy}{dx} = \frac{-14(1)-(1)}{(1)+14(1)}$$

$$\frac{dy}{dx} = \frac{-15}{15}$$

$$y - 1 = -1(x-1)$$

YOUR TURN

Solve the equation of the tangent line to the graph of $x^2 + 4y^2 = 4$ at the point $\left(\sqrt{2}, -\frac{1}{\sqrt{2}}\right)$.

$$y + \frac{1}{\sqrt{2}} = \frac{1}{2}\left(x - \sqrt{2}\right)$$

AP Multiple Choice Practice Question 1 (non-calculator)

If $5x^2 - 2xy + 7y^2 = 0$, then solve $\frac{dy}{dx}$

(A) $\frac{5x+7y}{x}$

(B) $\frac{y-5x}{7y}$

(C) $\frac{y-5x}{7y-x}$

(D) $10x - 2y + 14y$

AP Multiple Choice Practice Question 1 (non-calculator)

If $5x^2 - 2xy + 7y^2 = 0$, then solve $\frac{dy}{dx}$

Vocabulary	Connections and Process	Answer and Justifications
Derivative Implicit Diff. Power Rule Product Rule	$\frac{d}{dx}5x^2 = 10x \quad \frac{d}{dx}7y^2 = 14y\frac{dy}{dx}$ $\frac{d}{dx}2xy = 2x(1)\frac{dy}{dx} + y(2)\frac{dx}{dx}$ $10x - 2x\frac{dy}{dx} - 2y + 14y\frac{dy}{dx} = 0$ $-2x\frac{dy}{dx} + 14y\frac{dy}{dx} = -10x + 2y$ $\frac{dy}{dx}(-2x + 14y) = -10x + 2y$	$\frac{\frac{dy}{dx}(-2x+14y)}{-2x+14y} = \frac{-10x+2y}{-2x+14y}$ $\frac{2(-5x+y)}{2(-x+7y)} = \frac{y-5x}{7y-x}$ C

AP Multiple Choice Practice Question 2 (non-calculator)

If $\tan(x+y)=x$, then $\frac{dy}{dx}=$

(A) $\tan^2(x+y)$

(B) $\sec^2(x+y)$

(C) $\sin^2(x+y)-1$

(D) $\cos^2(x+y)-1$

AP Multiple Choice Practice Question 2 (non-calculator)

If $\tan(x+y) = x$, then $\frac{dy}{dx} =$

Vocabulary	Connections and Process	Answer and Justifications
Derivative Implicit Diff. Trig Derivative Power Rule	$\frac{d}{dx}\tan(x+y) =$ $\sec^2(x+y)\left(1+\frac{dy}{dx}\right) = 1$ $\frac{\sec^2(x+y)\left[1+\frac{dy}{dx}\right]}{\sec^2(x+y)} = \frac{1}{\sec^2(x+y)}$ $1+\frac{dy}{dx} = \frac{1}{\sec^2(x+y)}$	$1+\frac{dy}{dx} = \cos^2(x+y)$ $\frac{dy}{dx} = \cos^2(x+y) - 1$ *D*

ASSIGNMENT

Worksheet