CHAIN RULE

Section 2.4

Calculus BC AP/Dual, Revised ©2017

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REVIEW

Solve $g(f(x))$ if $f(x) = x^2 + 1$ and $g(x) = 6x^2$

$$g\left(f\left(x\right)\right)$$

$$g\left(x^2 + 1\right)$$

$$6\left(x^2 + 1\right)^2$$

$$6\left(x^2 + 1\right)^2$$

THE CHAIN RULE

A. $\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'(x)$

B. If $f(x)$ is a differentiable function and $g(x)$ is a differentiable function, then $y = f(g(x))$ is a differentiable function.

DEMONSTRATION

A. What happens when you want to eat a blow pop?

B. Order: Wrapper, Shell, Gum

C. Reach the shell first before reaching to the gum

Example 1

Solve $\frac{dy}{dx}$ for $y = \left(x^2 + 1\right)^3$

$x^2 + 1$ is the inside.

$$f'(x) = f'(g(x))g'(x)$$

Original	$\left(x^2+1\right)^3$	x^2+1
Derivative	$3\left(x^2+1\right)^2$	$2x$

$\left(x^2 + 1\right)^3$ is the shell.

$$f'(x) = 3\left(x^2+1\right)^2(2x)$$

$$f'(x) = 6x\left(x^2+1\right)^2$$

Example 2

Solve $\frac{dy}{dx}$ for $f(x) = \left(2x^2 + 5\right)^7$

$$f'(x) = 28x\left(2x^2 + 5\right)^6$$

EXAMPLE 3

Solve $\frac{dy}{dx}$ for $f(x) = \sqrt{x^2 + 1}$

$$f'(x) = \frac{x}{\left(x^2 + 1\right)^{1/2}}$$

YOUR TURN

Solve $\frac{dy}{dx}$ for $f(x) = \left(2x^3 + 1\right)^{10}$

$$f'(x) = 60x^2\left(2x^3+1\right)^9$$

Example 4

Solve $g'(t)$ for $g(t) = \frac{1}{3t^2+4}$

Could this work for the Quotient Rule?

$$g(t) = (3t^2+4)^{-1}$$

$(3t^2+4)^{-1}$	$3t^2+4$
$-1(3t^2+4)^{-2}$	$6t$

$$-1(3t^2+4)^{-2}(6t)$$

$$g'(t) = \frac{-6t}{(3t^2+4)^2}$$

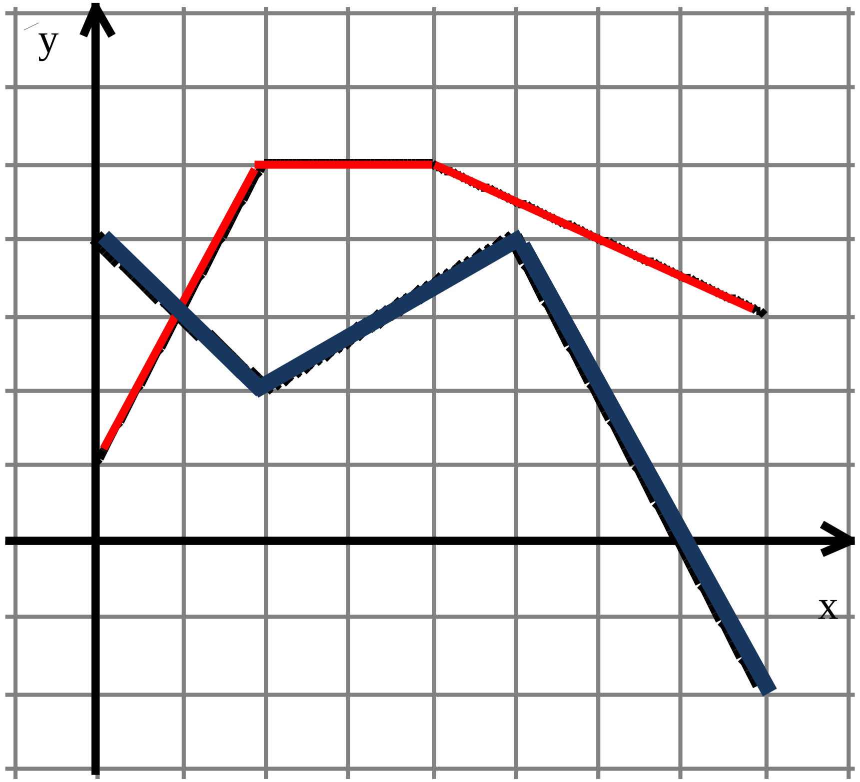
YOUR TURN

Solve $f'(x)$ for $f(x) = \frac{-7}{(2t-3)^2}$

$$f'(x) = \frac{28}{(2t-3)^3}$$

Example 5

Use the graphs of f (red graph) and g (blue graph) to solve for $h'(7)$ if $h(x) = g(f(x))$, if they exist.



$$h(x) = g(f(x))$$

$$h'(x) = g'(f(x))\,f'(x)$$

$$h'(7) = g'(f(7))\,f'(7)$$

$$h'(7) = g'(3.5)\,f'(7)$$

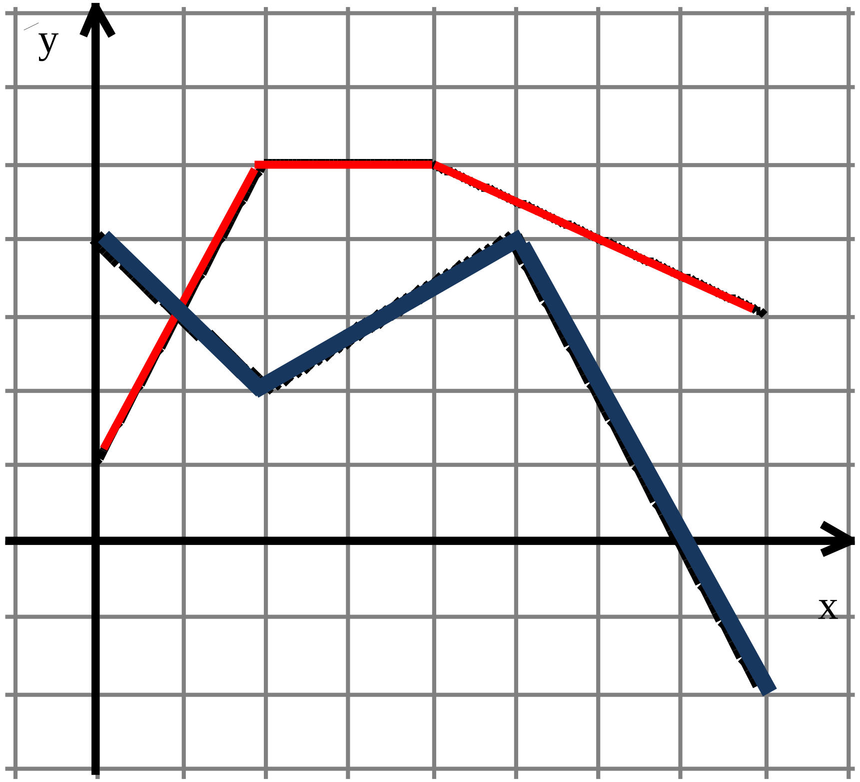
$$h'(7) = \left(\frac{g(4) - g(3)}{4 - 3}\right)\left(\frac{f(8) - f(6)}{8 - 6}\right)$$

$$h'(7) = \left(\frac{3.25 - 2.75}{1}\right)\left(\frac{3 - 4}{2}\right)$$

$$h'(7) = -\frac{1}{4}$$

Example 6

Use the graphs of f (red graph) and g (blue graph) to solve for $p'(6)$ if $p(x) = f(g(x))$, if they exist.



$$p(x) = f(g(x))$$

$$p'(x) = f'(g(x))g'(x)$$

$$p'(6) = f'(g(6))g'(6)$$

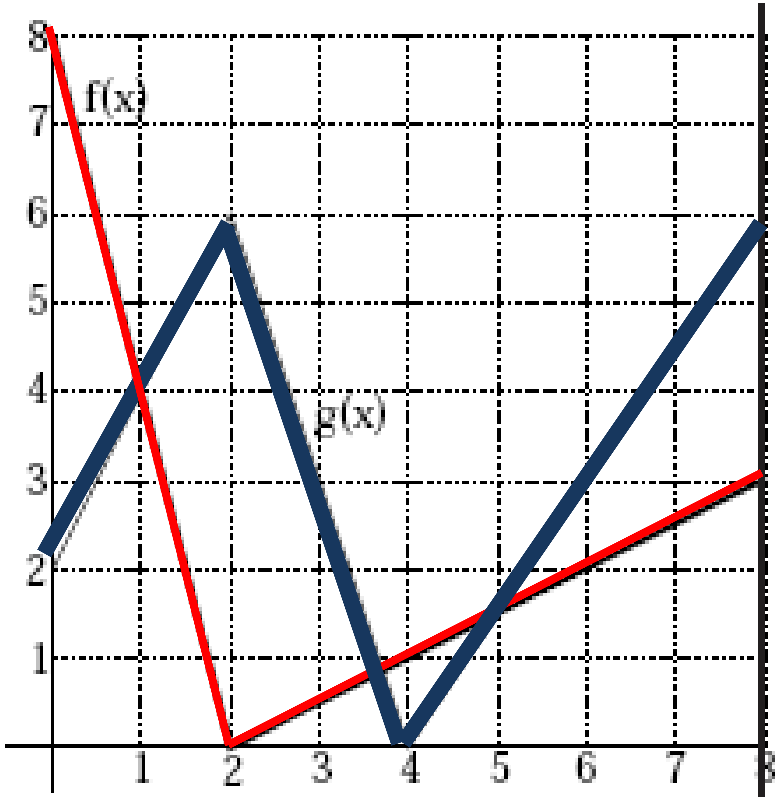
$$p'(6) = f'(2)g'(6)$$

$f'(2)$ is not differentiable because of a sharp turn

$$p'(6) = DNE$$

YOUR TURN

Use the graphs of f (red graph) and g (blue graph) to solve for $u'(6)$ if $u(x) = f(g(x))$, if they exist.



$$u'(6) = \frac{3}{4}$$

TRIG FUNCTION DERIVATIVES

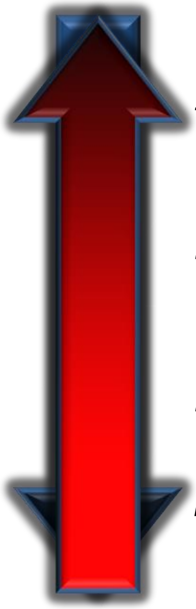
Derivatives of Trig Functions (MEMORIZE THEM)

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\csc x] = -\csc x \cot x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\cot x] = -\csc^2 x$

Trig Derivatives

So, when you have to ask your neighbor a question…

$$PSST\ldots$$



Positive

SEC

SEC

TAN

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

TRIG DERIVATIVES

Think opposite of PSST… is to CHANGE

Positive		*Negative*(*CHANGE*)
SEC		*CSC*
SEC	$\longrightarrow$	*CSC*
TAN		*COT*

- *CHANGE*
- *CSC*
- *CSC*
- *COT*

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

REVIEW

Basic Functions	Composite Functions
$\sin x$	$\sin(2x+3)$
e^x	e^{2x}
$\ln x$	$\ln(3x^4)$

Composite Functions

$$\sin\left(2x+3\right)$$

$$\left(\sin x\right)\left(2x+3\right))$$

EXAMPLE 7

Solve y' for $y = \sin(2x + 3)$

$$y' = f'(g(x))g'(x)$$

Original	$\sin(2x+3)$	$2x+3$
Derivative	$\cos(2x+3)$	2

$$y' = 2\cos(2x+3)$$

$$y' = 2\cos(2x+3)$$

EXAMPLE 8

Solve y' for $y = \tan^2 x$

$$y' = f'(g(x))g'(x)$$

Original	$(\tan x)^2$	$\tan x$	x
Derivative	$2(\tan x)$	$\sec^2(x)$	1

$$y' = 2(\tan x)(\sec^2 x)(1)$$

$$y' = 2\tan x \sec^2 x$$

EXAMPLE 9

Solve y' for $y = \csc\left(x^3\right)$

$$y' = -3x^2 \csc x^3 \cot x^3$$

YOUR TURN

Solve y' for $y = \cos{(3x)^2}$

$$y' = -18x\sin\left(9x^2\right)$$

WHEN YOU BUILD A CHAIN…

Revisit Order

A. **Wrapper**
B. **Shell**
C. **Gum**

WITH TRIG CHAIN RULE FUNCTIONS

A. Make sure the exponent and negative are put in the appropriate spots
B. Use parenthesis

Example 10

Solve $\frac{dy}{dx}$ for $y = \sin^3(4x)$

$$y' = f'(g(x))g'(x)$$

$(\sin(4x))^3$	$\sin(4x)$	$4x$
$3(\sin(4x))^2$	$\cos(4x)$	4

$$y' = 3(\sin^2 x)(\cos(4x))(4)$$

$$f'(x) = 12\sin^2(4x)\cos(4x)$$

EXAMPLE 11

Solve $\frac{dy}{dx}$ for $f(x) = 3\cos^3(x^3 + 1)$

$$f'(x) = -27x^2\cos^2\left(x^3+1\right)\sin\left(x^3+1\right)$$

YOUR TURN

Solve $\frac{dy}{dx}$ for $\boldsymbol{f(x) = 4\sec^3(2x+1)}$

$$f'(x) = 24\sec^3(2x+1)\tan(2x+1)$$

Example 12

Solve $f''(x)$ for $f(x) = x^4 \sin x$

$$y = x^4 \sin x$$

$$y' = 4x^3 \sin x + x^4 \cos x$$

$$y'' = fg' + gf'$$

$$4x^3 \cos x + \sin x\left(12x^2\right) + \left(4x^3\right)\cos x + x^4\left(-\sin x\right)$$

$$f''\left(x\right) = 12x^2 \sin x + 8x^3 \cos x - x^4 \sin x$$

YOUR TURN

Solve $f'(x)$**for** $f(x) = \frac{-2x^2-5}{\cos(2x^3)}$

$$f'(x) = \frac{-2x\left(2\cos 2x^3 + 6x^3\sin 2x^3 - 15x\sin 2x^3\right)}{\left(\cos^2 2x^3\right)}$$

EXAMPLE 13

If $y = \tan u$, $u = v - \frac{1}{v}$, and $v = \ln x$ what is the value of $\frac{dy}{dx}$ at $x = e$?

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du}(\tan u) \cdot \frac{du}{dv}\left(v - \frac{1}{v}\right) \cdot \frac{dv}{dx}(\ln x)$$

$$\frac{dy}{dx} = \left(\sec^2(0)\right) \cdot \left(1 + \frac{1}{1^2}\right) \cdot \left(\frac{1}{e}\right)$$

$$\frac{2}{e}$$

$$x = e$$

$$v = \ln e = 1$$

$$u = 1 - \frac{1}{1} = 1 - 1 = 0$$

EXAMPLE 14

Determine the point(s) in the interval of $(0, 2\pi)$ at which the graph of $f(x) = \frac{\cos x}{2+\sin x}$ has a horizontal tangent.

$$f(x) = \frac{\cos x}{2+\sin x}$$

$$f'(x) = \frac{(2+\sin x)(-\sin x) - (\cos x)(\cos x)}{(2+\sin x)^2}$$

$$f'(x) = \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}$$

$$f'(x) = \frac{-2\sin x - (\sin^2 x + \cos^2 x)}{(2+\sin x)^2} = \frac{\boxed{-2\sin x - 1}}{(2+\sin x)^2}$$

EXAMPLE 14

Determine the point(s) in the interval of $(0, 2\pi)$ at which the graph of $f(x) = \frac{\cos x}{2+\sin x}$ has a horizontal tangent.

$$f'(x) = \frac{-2\sin x - 1}{(2+\sin x)^2}$$

$$-2\sin x - 1 = 0$$

$$-2\sin x = 1$$

$$\sin x = -\frac{1}{2}$$

$$\left(\frac{7\pi}{6}, -\frac{\sqrt{3}}{3}\right), \left(\frac{11\pi}{6}, \frac{\sqrt{3}}{3}\right)$$

Your Turn

Determine the equation of the tangent line using the equation, $y = \frac{1}{x} + \sqrt{\cos x}$ at $(\pi, 1)$.

$$y = x^{-1} + (\cos x)^{1/2}$$

$$y' = -x^{-2} + \frac{1}{2}(\cos x)^{-1/2}(-\sin x)$$

$$y' = -\frac{1}{x^2} - \frac{\sin x}{2\sqrt{\cos x}}$$

$$y'(\pi) = -\frac{1}{(\pi)^2} - \frac{\sin(\pi)}{2\sqrt{\cos(\pi)}}$$

YOUR TURN

Determine the equation of the tangent line using the equation, $\boldsymbol{y = \frac{1}{x} + \sqrt{\cos x}}$ at $\boldsymbol{(\pi, 1)}$.

$$y'(\pi) = -\frac{1}{(\pi)^2} - \frac{\sin(\pi)}{2\sqrt{\cos(\pi)}}$$

$$y'(\pi) = -\frac{1}{\pi^2} - \frac{0}{2\sqrt{1}}$$

$$y'(\pi) = -\frac{1}{\pi^2}$$

$$y - 1 = \left(-\frac{1}{\pi^2}\right)(x - \pi)$$

AP Multiple Choice Practice Question 1 (non-calculator)

Solve the derivative of $s(t) = \sec\sqrt{t}$

(A) $\sec\frac{1}{2\sqrt{t}}\ \tan\frac{1}{2\sqrt{t}}$

(B) $\sec\sqrt{t}\ \tan\sqrt{t}$

(C) $\frac{\sec\sqrt{t}\tan\sqrt{t}}{2\sqrt{t}}$

(D) $\tan^2\sqrt{t}$

AP Multiple Choice Practice Question 1 (non-calculator)

Solve the derivative of $s(t) = \sec\sqrt{t}$

Vocabulary	Connections and Process	Answer and Justifications
Derivative Chain Rule Trigonometric Derivative	$\frac{d}{dt}\sec t = \sec t \tan t$ $\frac{d}{dt}\sqrt{t} = \frac{d}{dt}\left(t^{1/2}\right) = \frac{1}{2}t^{-1/2}$ $\frac{d}{dt}\sec\sqrt{t} = \frac{1}{2}\left(\sec\left(t^{1/2}\right)\left(\tan\left(t^{1/2}\right)\right)\right)t^{-1/2}$ $\frac{d}{dt}\sec\sqrt{t} = \frac{\sec\sqrt{t}\tan\sqrt{t}}{2\sqrt{t}}$	C

AP Multiple Choice Practice Question 2 (non-calculator)

Solve the derivative of $f(\theta) = \sqrt{\sin 2\theta}$

(A) $\dfrac{\cos 2\theta}{\sqrt{\sin 2\theta}}$

(B) $\dfrac{\cos 2\theta}{2\sqrt{\sin 2\theta}}$

(C) $\cos 2\theta$

(D) $\sqrt{\sec 2\theta}$

AP Multiple Choice Practice Question 2 (non-calculator)

Solve the derivative of $f(\theta) = \sqrt{\sin 2\theta}$

Vocabulary	Connections and Process	Answer and Justifications
Derivative Chain Rule Trigonometric Derivative	$\frac{d}{d\theta}\sin 2\theta = \cos(2\theta)\cdot 2$ $\frac{d}{d\theta}(\sin 2\theta)^{1/2} = \frac{1}{2}(\sin 2\theta)^{-1/2}(\cos(2\theta)\cdot 2)$ $\frac{d}{d\theta}(\sin 2\theta)^{1/2} = \frac{\cos 2\theta}{\sqrt{\sin 2\theta}}$	A

ASSIGNMENT

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7-31 odd, 43-57 odd, 65-71 odd, 84, 85, 87, 102