PRODUCT AND QUOTIENT RULE

Section 2.3

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Guess The Rule

Equation	Derivative (not being simplified)
1) $\frac{d}{dx}\left[\left(2x^2\right)(5+4x)\right]$	$\left(2x^2\right)(4)+(5+4x)(4x)$
2) $\frac{d}{dx}\left[\left(3x-2x^2\right)(5+4x)\right]$	$\left(3x-2x^2\right)(4)+(5+4x)(3-4x)$
3) $\frac{d}{dx}\left[\left(\sqrt{x}\right)(x^2+2)\right]$	$\left(x^{1/2}\right)(2x)+\left(x^2+2\right)\left(\frac{1}{2x^{1/2}}\right)$
4) $\frac{d}{dx}\left[x^2 \sin x \cos x\right]$	$2x\sin x\cos x+x^2\cos x\cos x+x^2\sin x(-\sin x)$
5) $\frac{d}{dx}\left[f(x)g(x)\right]$	$f(x)g'(x)+g(x)f'(x)$

The Product Rule

“First d-second plus second d-first”

“First d-second plus second d-first”

“First d-second plus second d-first”

The Product Rule

A. $cf(x)g'(x) + g(x)f'(x)$

B. Many times, there can be more than one derivative can be taken for a differentiable function. These derivatives imply continued continuity (like the first derivative):

1. First Derivative: $y' = f'(x) = \frac{dy}{dx}$
2. Second Derivative: $y'' = f''(x) = \frac{d^2y}{dx^2}$
3. Third Derivative: $y''' = f'''(x) = \frac{d^3y}{dx^3}$
4. Fourth Derivative: $y^{(4)} = f^{(4)}(x) = \frac{d^4y}{dx^4}$
5. nth Derivative: $y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n}$

Proof of Product Rule

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x) + f(x+h)g(x) - f(x+h)g(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

Proof of Product Rule

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0} \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right]}{h} + \lim_{h\to 0} \frac{g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$\underbrace{\left(\lim_{h\to 0} f(x+h)\right)}_{f(x)} \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} + \underbrace{\left(\lim_{h\to 0} g(x)\right)}_{g(x)} \underbrace{\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}_{f'(x)}$$

$$f(x)g'(x) + g(x)f'(x)$$

Example 1

Solve for the derivative of $f(x) = x^2(2x - 3)$ (don't multiply them)

$$f'(x) = f(x)g'(x) + g(x)f'(x)$$

$$f'(x) = (x^2)\frac{d}{dx}(2x-3) + (2x-3)\frac{d}{dx}(x^2)$$

$$f'(x) = (x^2)(2) + (2x-3)(2x)$$

$$f'(x) = 2x^2 + 4x^2 - 6x$$

$$f'(x) = 6x^2 - 6x$$

YOUR TURN

Solve for the derivative of $f(x) = (-2x^4 + 5x^2 + 4)(-3x^2 + 2)$

$$f'(x) = 36x^5 - 76x^3 - 4x$$

EXAMPLE 2

Solve for the derivative of $f(x) = 3x^3 \sin x$

$$f'(x) = f(x)g'(x) + g(x)f'(x)$$

$$f'(x) = (3x^3)\frac{d}{dx}(\sin x) + (\sin x)\frac{d}{dx}(3x^3)$$

$$f'(x) = (3x^3)(\cos x) + (\sin x)(9x^2)$$

$$f'(x) = 3x^3 \cos x + 9x^2 \sin x$$

YOUR TURN

Solve for the derivative of $f(x) = 2x \sin x$ by using only the Product Rule

$$f'(x) = 2\sin x + 2x\cos x$$

ACCELERATION

A. **Acceleration can be defined as the second derivative of the position function or the first derivative of the velocity of the function**

B. **Rate of Change of Velocity**

C. $\boldsymbol{f(t) = v'(t) = a''(t)}$

$$Position = f\left(t\right)$$

$$Velocity = f\,'\left(t\right) = v\left(t\right)$$

$$Acceleration = f\,''\left(t\right) = v\,'\left(t\right) = a\left(t\right)$$

Example 3

The position of an object is defined by the equation, $y = \frac{1}{4}t^4 + t^2$. What is the acceleration of the object at $t = 2$?

$$f(t) = s(t)$$

$$f'(t) = v(t)$$

$$f''(t) = a(t)$$

$$f(t) = \frac{1}{4}t^4 + t^2$$

$$f'(t) = t^3 + 2t$$

$$f''(t) = 3t^2 + 2$$

$$f''(2) = 3(2)^2 + 2$$

$$f''(2) = 14$$

EXAMPLE 4

Given $\frac{dy}{dx} = 5x^4 - 3x$, solve for $\frac{d^4y}{dx^4}$.

$$\frac{d^4y}{dx^4} = 120x$$

YOUR TURN

Given $f''(x) = 6x + 27x^{-2}$, solve for $f^{(4)}(x)$

$$f^{(4)}(x) = \frac{162}{x^4}$$

Guess The Rule

Equation	Derivative (not being simplified)
1) $\frac{d}{dx}\left[\frac{2}{x^2+1}\right]$	$\frac{(x^2+1)(0)-(2)(2x)}{(x^2+1)^2}$
2) $\frac{d}{dx}\left[\frac{5x-2}{x^2+1}\right]$	$\frac{(x^2+1)(5)-(5x-2)(2x)}{(x^2+1)^2}$
3) $\frac{d}{dx}\left[\frac{1-\cos x}{\sin x}\right]$	$\frac{(\sin x)(\sin x)-(1-\cos x)(\cos x)}{\cos^2 x}$
4) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]$	$\frac{g(x)f'(x)-f(x)g'(x)}{\left[g(x)\right]^2}$

The Quotient Rule

"Low D-High minus High D-Low over Low Times Low"

"Low D-High minus High D-Low over Low Times Low"

"Low D-High minus High D-Low over Low Times Low"

THE QUOTIENT RULE

A. $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{\left[g(x)^2\right]}$

THE QUOTIENT RULE PROOF

Proof As with the proof of Theorem 2.7, the key to this proof is subtracting and adding the same quantity.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \lim_{\Delta x \to 0} \frac{\dfrac{f(x+\Delta x)}{g(x+\Delta x)} - \dfrac{f(x)}{g(x)}}{\Delta x} \qquad \text{Definition of derivative}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{g(x)f(x+\Delta x) - f(x)g(x) + f(x)g(x) - f(x)g(x+\Delta x)}{\Delta x g(x)g(x+\Delta x)}$$

$$= \frac{\displaystyle\lim_{\Delta x \to 0} \frac{g(x)[f(x+\Delta x) - f(x)]}{\Delta x} - \lim_{\Delta x \to 0} \frac{f(x)[g(x+\Delta x) - g(x)]}{\Delta x}}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

$$= \frac{g(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - f(x)\left[\displaystyle\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}\right]}{\displaystyle\lim_{\Delta x \to 0} [g(x)g(x+\Delta x)]}$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

EXAMPLE 5

Solve for the derivative of $f(x) = \frac{x^2+2x}{x}$

$$f'(x) = \frac{g(x)\,f'(x) - f(x)\,g'(x)}{\left[g(x)\right]^2}$$

$$f'(x) = \frac{x\frac{d}{dx}\left(x^2+2x\right) - \left(x^2+2x\right)\frac{d}{dx}(x)}{(x)^2}$$

$$f'(x) = \frac{x(2x+2) - \left(x^2+2x\right)(1)}{(x)^2}$$

EXAMPLE 5

Solve for the derivative of $\boldsymbol{f(x) = \frac{x^2+2x}{x}}$

$$f'(x) = \frac{x(2x+2)-(x^2+2x)(1)}{(x)^2}$$

$$f'(x) = \frac{2x^2+2x-x^2-2x}{x^2}$$

$$f'(x) = 1$$

EXAMPLE 6

Solve for the derivative of $f(x) = \frac{3-\left(\frac{1}{x}\right)}{x+5}$

$$f'(x) = \frac{-3x^2 + 2x + 5}{x^2\left(x+5\right)^2}$$

EXAMPLE 7

Solve $\frac{d^2y}{dx^2}$ for $f(x) = \frac{x-1}{x+2}$

$$f'(x) = \frac{-6}{(x+2)^3}$$

YOUR TURN

Solve $\frac{dy}{dx}$ for $f(x) = \frac{6x+7}{7-2x}$

$$f'(x) = \frac{56}{(7-2x)^2}$$

REVIEW

A. Trig Functions

1. $\frac{1}{\sin x} = \csc x$
2. $\frac{1}{\cos x} = \sec x$
3. $\frac{1}{\tan x} = \cot x$
4. $\cot x = \frac{\cos x}{\sin x} = \frac{\csc x}{\sec x}$

B. Pythagorean Identities

1. $\sin^2 x + \cos^2 x = 1$
2. $1 + \tan^2 x = \sec^2 x$
3. $1 + \cot^2 x = \csc^2 x$

Trig Function Derivatives

C. Derivatives of Trig Functions (MEMORIZE THEM)

1. $\frac{d}{dx}[\sin x] = \cos x$
2. $\frac{d}{dx}[\cos x] = -\sin x$
3. $\frac{d}{dx}[\tan x] = \sec^2 x$
4. $\frac{d}{dx}[\csc x] = -\csc x \cot x$
5. $\frac{d}{dx}[\sec x] = \sec x \tan x$
6. $\frac{d}{dx}[\cot x] = -\csc^2 x$

PROOF OF $\frac{d}{dx}[\tan x]$

$$f(x) = \frac{\sin x}{\cos x}$$

$$f'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$f'(x) = \frac{\cos(x)\cos(x) - -\sin(x)\sin(x)}{[\cos^2 x]}$$

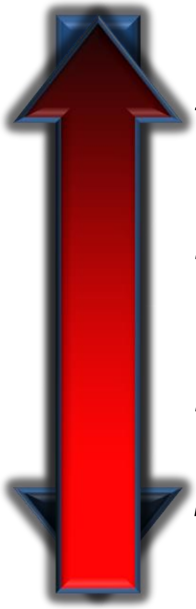
$$f'(x) = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \qquad f'(x) = \frac{1}{\cos^2 x}$$

$$f'(\tan x) = \sec^2 x$$

Trig Derivatives

So, when you have to ask your neighbor a question…

$$PSST\ldots$$



Positive

SEC

SEC

TAN

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

TRIG DERIVATIVES

Think opposite of PSST… you "*c*"

Positive		*Negative*(*CHANGE*)
SEC		*CSC*
SEC	$\longrightarrow$	*CSC*
TAN		*COT*

CHANGE

CSC

CSC

COT

$$\frac{d}{dx}\left[\csc x\right] = -\csc x \cot x$$

$$\frac{d}{dx}\left[\cot x\right] = -\csc^2 x$$

Example 8

Solve for the derivative of $f(x) = \sin x \tan x$

$$f'(x) = f(x)g'(x) + g(x)f'(x)$$

$$f'(x) = (\sin x)\frac{d}{dx}(\tan x) + (\tan x)\frac{d}{dx}(\sin x)$$

$$f'(x) = (\sin x)\left(\sec^2 x\right) + (\tan x)(\cos x)$$

$$f'(x) = (\sin x)\left(\sec^2 x\right) + (\tan x)(\cos x)$$

EXAMPLE 9

Solve for the derivative of $\boldsymbol{f(x) = \frac{\cos x}{1+\sin x}}$

$$f'(x) = -\frac{1}{(1+\sin x)}$$

YOUR TURN

Solve for the derivative of $f(x) = \frac{\sec x}{x}$ by using the Quotient Rule

$$f'(x) = \frac{\sec x\left(x\tan x - 1\right)}{x^2}$$

Example 10

Using the table given $h(x) = f(x)g(x)$, solve for $h'(2)$

x	2	3
$f(x)$	1	-3
$f'(x)$	-2	1
$g(x)$	3	4
$g'(x)$	6	2

$$h'(2) = f(x)g'(x) + g(x)f'(x)$$

$$h'(2) = f(2)g'(2) + g(2)f'(2)$$

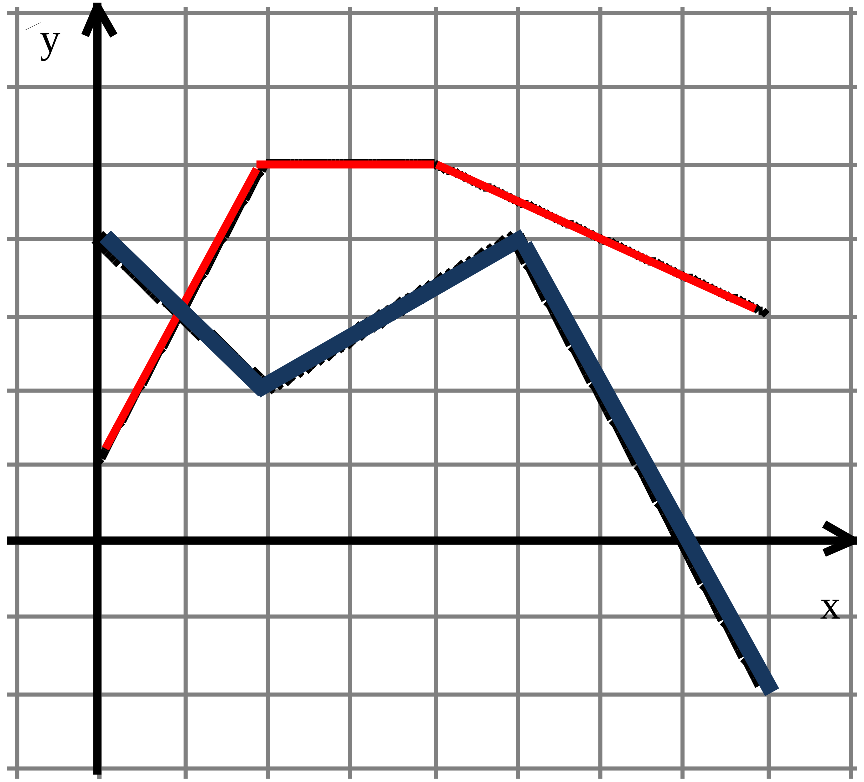
$$h'(2) = (1)(6) + (3)(-2)$$

$$h'(2) = 6 + -6$$

$$h'(2) = 0$$

Example 11

Use the graphs of f (red graph) and g (blue graph) to solve for $h'(7)$ if $h(x) = g(x)f(x)$, if they exist.



$$h(x) = g(x)f(x)$$

$$h'(x) = g(x)f'(x) + f(x)g'(x)$$

$$h'(7) = g(7)f'(7) + f(7)g'(7)$$

$$h'(7) = (0)\left(-\frac{1}{2}\right) + \left(\frac{7}{2}\right)(-2)$$

$$h'(7) = 0 + (-7)$$

$$h'(7) = -7$$

Your Turn

Using the table given $q(x) = \frac{f(x)}{g(x)}$, solve for $q'(3)$

x	2	3
$f(x)$	1	-3
$f'(x)$	-2	1
$g(x)$	3	4
$g'(x)$	6	2

$$q'(3) = \frac{5}{8}$$

AP Multiple Choice Practice Question 1 (non-calculator)

If $f''(x) = 3x^2 + 6x + 4$, solve for $f^{(4)}(x)$

(A) 0

(B) 6

(C) $2x + 6$

(D) $6x + 6$

AP Multiple Choice Practice Question 1 (non-calculator)

If $f''(x) = 3x^2 + 6x + 4$, solve for $f^{(4)}(x)$

Vocabulary	Connections and Process	Answer and Justifications
4th Derivative Power Rule	$f''(x) = 3x^2 + 6x + 4$ $f'''(x) = 6x + 6$ $f^{4}(x) = 6$	**B** The 4th derivative of the function is 6 when the 2nd derivative is $3x^2 + 6x + 4$

AP Multiple Choice Practice Question 2 (non-calculator)

Find the derivative of $x^2\ f(x)$.

(A) $x\,[x\ f'(x) + 2\ f(x)]$

(B) $2x\ f'(x)$

(C) $x\,[x\ f(x) + 2\ f'(x)]$

(D) $x^2 f'(x)$

AP Multiple Choice Practice Question 2 (non-calculator)

Find the derivative of $x^2\, f(x)$.

Vocabulary	Connections and Process	Answer and Justifications
Derivative Product Rule	$f'(x) = fg' + gf'$ $f = x^2, g = f(x)$ $(x^2)(f'(x)) + (f(x))(2x)$ $x[(x)(f'(x)) + (f(x))(2)]$	A

AP Multiple Choice Practice Question 3 (non-calculator)

Find $\frac{d}{d\theta}$ for $y = \csc\theta - \cot\theta$

(A) $-\csc\theta\cot\theta + \csc^2\theta$

(B) 0

(C) $-\cot^2\theta + \csc\theta\cot\theta$

(D) $\sec\theta\tan\theta - \sec^2\theta$

AP Multiple Choice Practice Question 3 (non-calculator)

Find $\frac{d}{d\theta}$ for $y = \csc\theta - \cot\theta$

Vocabulary	Connections and Process	Answer and Justifications
Derivative Trig Derivatives	$\frac{d}{dx}\csc\theta = -\csc\theta\cot\theta$ $\frac{d}{dx}\cot\theta = -\csc^2\theta$ $\frac{d}{d\theta} = -\csc\theta\cot\theta - \left(-\csc^2\theta\right)$	A

Assignment

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1-12 all, 13-17 odd, 25-31 odd, 39-43 odd, 47-51 odd, 63, 65, 67A, 73, 75, 91-97 odd