§1.3: Evaluating Limits Analytically, §1.4: One-Sided Limits, and §1.4a: Continuity
"I WILL…
…Solve a limit through different processes, evaluate one-sided limits, and imply continuity rules"

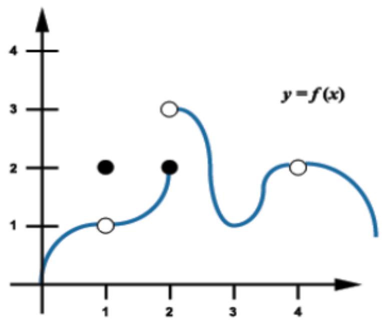
I. Properties of Limits

A. Let *b* and *c* be real numbers, let *n* be a positive integer, and let *f* and *g* be functions with the given limits: $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$

1. Scalar Multiple: $\lim_{x\to c} [bf(x)] = b \lim_{x\to c} [f(x)]$
2. Sum or Difference: $\lim_{x\to c} [f(x) \pm g(x)] = \lim_{x\to c} [f(x)] \pm \lim_{x\to c} [g(x)]$
3. Product: $\lim_{x\to c} [f(x)g(x)] = \lim_{x\to c} [f(x)] \cdot \lim_{x\to c} [g(x)]$
4. Quotient: $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{\lim_{x\to c} [f(x)]}{\lim_{x\to c} [g(x)]}$

Ex 1: Find the following limit given $\lim_{x\to c} f(x) = \frac{7}{6}$ and $\lim_{x\to c} f(x) = \frac{5}{6}$ determine $\lim_{x\to c} f(x) + \lim_{x\to c} g(x)$

Ex 2: Given the graph of $f(x)$, find $\frac{\lim_{x\to 1}(5f(x)+f(1))}{\lim_{x\to 4}(f(x))^2}$



Your Turn: Solve the following limit given $\lim_{x\to -1} f(x) = 3x^2 - 2x - 1$ and $\lim_{x\to -1} g(x) = x^2 + 1$, determine $\lim_{x\to -1} h(x) = \frac{f(x)}{g(x)}$

II. "0/0" Limits AKA: Indeterminate Form

A. Always begin with direct substitution
B. Completely factor the problem
C. Simplify and/or Cancel by identifying a function g that agrees with for all *x* except $x = c$. Take the limit of g
D. Apply algebra rules
   1. If necessary, Rationalize the numerator or denominator
   2. Plug in x of the function to get the limit

Ex 3: Evaluate $\lim_{x\to -1}\frac{x^2-16}{x-4}$	Your Turn: Evaluate $\lim_{x\to -3}\frac{\sqrt{x+7}-2}{x+3}$
Ex 4: Evaluate $\lim_{x\to 0}\frac{\frac{1}{5+x}-\frac{1}{5}}{x}$	Your Turn: Solve $\lim_{x\to a}\frac{x-a}{x^3-a^3}$

III. Squeeze Theorem

A. Also known as the "Sandwich theorem," it is used to evaluate the limit of a function that can't be computed at a given point.

B. For a given interval containing point c, where f, g, and h are three functions that are differentiable and $g(x) < f(x) < h(x)$ over the interval where $f(x)$ is the upper bound and $h(x)$ is the lower bound.

Ex 5: Use the Squeeze Theorem to evaluate $\lim_{x\to c} g(x)$ where $c = 1$ for $3x \le g(x) \le x^3 + 2$

IV. Special Trigonometric Limits

A. $\lim_{x\to 0} \frac{\sin x}{x} = 1$

B. $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$

C. $\lim_{x\to 0} (1+x)^{1/x} = e$

D. When expressing x in radians and not in degrees

Ex 6: Evaluate $\lim_{x\to 0} \frac{\tan(x)}{x}$

Ex 7: Evaluate $\lim_{x\to 0} \frac{\sin(4x)}{x}$

Your Turn: $\lim_{x\to 0} \frac{5\sin(x)}{3x}$

V. One-Sided Limits

A. Plug the x to the limit

1. When a number is plugged into the equation and results to a number being divided by zero.

B. Define the left and right side

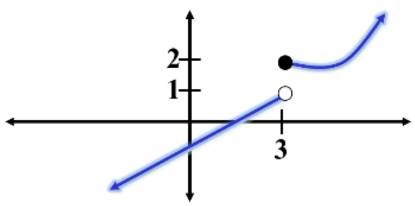
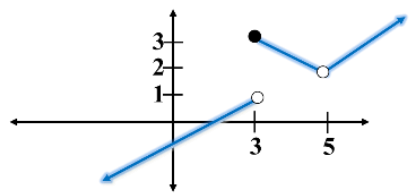
1. $x \to c^-$ means the numbers on the left side by plugging a number LESSER than the limit
2. $x \to c^+$ means the numbers on the right side by plugging a number GREATER than the limit
3. One sided limits can occur where normal limits do not
4. Let f be a function and let c and L be real numbers. The limit of $f(x)$ as x approaches c is L if and only if: $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x)$

C. Compare left and right side

1. If the left side is the same as the right side, the LIMIT EXISTS with the answer
2. If the left side is different as the right side, the LIMIT DOES NOT EXIST

D. Dividing Numbers

1. Number/Small ± Number = BIG POSITIVE/NEGATIVE NUMBER = ∞
2. Number/Big ± Number = SMALL POSITIVE/NEGATIVE NUMBER = –∞

Ex 8: Given the graph below, evaluate $\lim_{x\to 3} f(x)$	Your Turn: Given the graph below, evaluate $\lim_{x\to 3} f(x)$ and $\lim_{x\to 5} f(x)$
 	

VI. Discontinuity

A. Removable Discontinuity is a discontinuity where limits exist. Holes are an example.

B. Non-removable discontinuity is where the graph is not connected and cannot be made connected simply by filling in a single point.

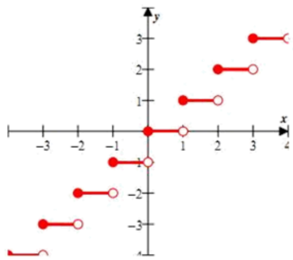
1. Jump
2. Infinite
3. Oscillating

Ex 9: Given the graph, $f(x) = [\![x]\!]$, determine $\lim_{x\to 0}[\![x]\!]$

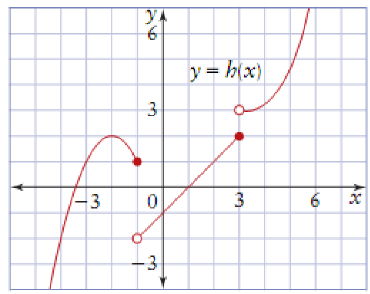
$\lim_{x\to 0^-} x =$

$\lim_{x\to 0^+} x =$

$\lim_{x\to 0} x =$



Your Turn: Given the graph $h(x)$ below, determine $\lim_{x\to 3^-} f(x)$, $\lim_{x\to 3^+} f(x)$, and $\lim_{x\to 3} f(x)$.

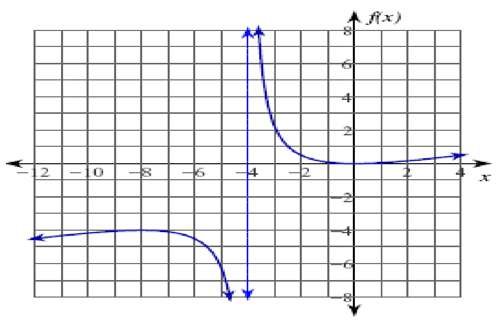


$(A) \lim_{x\to 3^-} h(x) =$

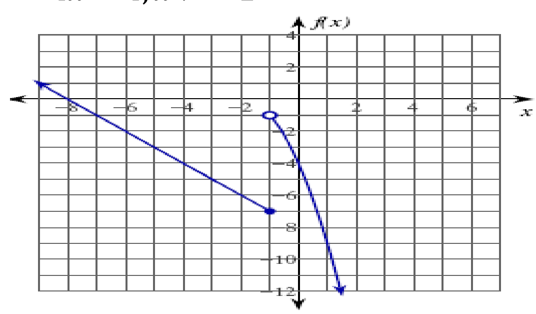
$(B) \lim_{x\to 3^+} h(x) =$

$(C) \lim_{x\to 3} h(x) =$

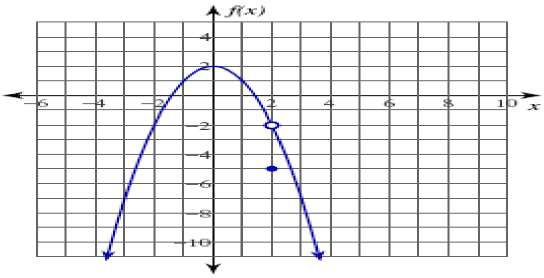
Ex 10: Identify all discontinuities of the graph below by establishing the undefined values of $f(x) = \frac{x^2}{4x+16}$



Ex 11: Identify all discontinuities of the graph below by establishing the undefined values of $f(x) = \begin{cases} -x-8, x \le -1 \\ -x^2-4x-4, x > -1 \end{cases}$



Your Turn: Identify all discontinuities of the graph below by establishing the undefined values of $f(x) = \begin{cases} -x^2+2, x \neq 2 \\ -5, x = 2 \end{cases}$



Ex 12: Solve for the values of a and b that makes $f(x)$ continuous for the function, $f(x) = \begin{cases} ax+3 & if \ x < 5 \\ 8 & if \ 5 \leq x < 10 \\ x^2+bx+1 & if \ x \geq 6 \end{cases}$

Your Turn: Solve for the values of a and b that makes $f(x)$ continuous, $f(x) = \begin{cases} \frac{x^2-4}{x-2} & if \ x < 2 \\ ax^2 - bx + 3 & if \ 2 \leq x < 3 \\ 2x - a + b & if \ x \geq 3 \end{cases}$

VII. Continuity on a Closed Interval

A. A function is continuous on the closed interval $[a,\ b]$ if it is continuous on the open interval (a, b) and if $\lim_{x \to a^+} f(a)$ and $\lim_{x \to b^-} f(b)$.

B. The function f is continuous from the right at a and continuous from the left at b.

VIII. Properties of Continuity

A. A function is continuous at the point $x = c$ if and only if:

1. $f(c)$ is defined
2. $\lim_{x \to c} f(x)$ exists
3. $\lim_{x \to c} f(x) = f(c)$

B. If the functions f and g are continuous at $x = c$, then the following are also continuous at c (just at a certain point, not everywhere).

C. Types:

1. Scalar Multiple: $b \bullet f$
2. Sum and Difference: $f \pm g$
3. Product: fg
4. Quotient: $\frac{f}{g}$ if $g(c) \neq 0$

Ex 13: Evaluate $\lim_{x\to 0}\frac{1}{x}$ and identify any x–values which are removable and non–removable discontinuity.	Ex 14: Solve $\lim_{x\to 4^-}\frac{x-4}{x^2-16}$
Ex 15: Solve $\lim_{x\to 0^+}\frac{\lvert x\rvert}{x}$	Ex 16: Solve $\lim_{x\to 2^+}(2x-[\![x]\!])$. If it does not exist, explain why.

Your Turn: Solve $\lim_{x \to 6^-}(4x - [\![x]\!])$. If it does not exist, explain why.

AP 1) Solve $\lim_{x \to \pi/2} \frac{\sin x}{x}$

(A) 0 (B) $-\pi/2$ (C) $(2\sqrt{2})/\pi$ (D) $2/\pi$

Vocabulary	Connections and Process	Answer and Justifications

AP 2) Given the following table of values below. Solve for $\lim_{x \to 1^+} f(x)$

x	0.7	0.8	0.9	0.99	1	1.01	1.1	1.2	1.3
$f(x)$	1.67	1.76	1.91	1.98	5	4.03	4.12	4.23	4.36

(A) 2 (B) 4 (C) 5 (D) Does Not Exist

Vocabulary	Connections and Process	Answer and Justifications

Page 67: 7, 13, 21, 25, 27-37 odd, 45-57 odd, 63-73 odd, 89
Page 78: 1-13 EOO, 19-25 odd, 27, 29, For 35-45 EOO, 49, 53, 59: list whether these functions have removable or non-removable discontinuities and solve), 61-65 odd